Welcome to my Map Projection Pages

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An introduction to cartography emphasizing *map projections*: their properties, applications and basic mathematics.

**Cartographical Map Projections**

South America in selected projections at identical scale. Which projection is best? Which is right? The short answer is none, at least not all the time. Even if a single projection is used, just switching the aspect can also radically reshape the continents.

**Cartography** is the science of map-making. It comprises many problems and techniques, including:
• measuring Earth's shape and features
• collecting and storing information about terrain, places and people
• adapting three-dimensional features to flat models (my main concern)
• devising and designing conventions for graphical representation of data
• printing and publishing information.

There is an endless variety of geographical maps for every kind of purpose. When looking at two different world maps one can wonder why the differences: do we draw the world as a rectangle, or an oval? Shouldn't it be a circle? Should grid lines be parallel, straight or curved? Does South America's "tail" bend eastwards or westwards? What's the "right" way (or, more properly, *is there one?*) to draw our unique planet?

One important concern of cartography is solving how to *project*, i.e. transform or map points from an almost spherical lump of rock (our Earth) onto either flat sheets of paper or not-so flat phosphorus-coated glass.

Here are informally described important cartographic concepts, how maps are drawn and why there are so many different kinds of projections for world maps. You may start reading [here](#) and follow the buttons, or use this table of contents:

### Introduction
- A gentle *introduction* to tinkering with maps
- Basic definitions and concepts about the Earth and maps

### Fitting Map to Purpose
- Useful map *properties*:
  - preserving *distances* (equidistance, isometry)
  - *geodesics* (great circles)
  - preserving *directions*
  - preserving *shape* (conformality)
  - preserving *area* (equivalence)
  - general *distortion pattern* (Tissot indicatrices)

### Mathematics of Cartography
- How projections are created, including equations for:
  - *azimuthal orthographic*
  - *stereographic cylindrical*
  - *sinusoidal (Sanson-Flamsteed)*
  - Mollweide
  - *polar/equatorial azimuthal equidistant/equal-area*
  - *equidistant cylindrical/Winkel I and II*
  - Aitoff, Hammer and Winkel Tripel

### Main Projection Groups
- *Azimuthal* projections, or piercing the Earth with laser beams
Introduction

An Interest in Map Projections

A cartographic projection is a transformation (also called mapping) from a round surface to a plane. There are many different projections, since there are several interesting or useful properties to fulfill. For instance, it would be desirable to keep shape, distance and area relationships exactly as in the original surface. Unfortunately, it can proved that there is not and there will never be such a perfect projection: every one is bound to distort at least part of the mapped region.

So cartography is an art and science of trade-offs and guidelines for designing and choosing the least inappropriate projection for each purpose.

I have always liked playing with world maps and wondered how computers could be used for mapping. I spent a good time deducing
formulas for projecting radius, latitude and longitude into cartesian $x$ and $y$. Of course, I could only draw coordinate grids until the day I got a public-domain database of geographical coordinates (at first my PC-XT spent over one hour to draw a rough map).

One of my renditions of Eckert's projection IV

My Own Projection
After some time I started devising my "own" projections (actually I had little access to map bibliography, so I could have reinvented the wheel). This projection of mine was inspired by both Sanson's, Flamsteed's and Eckert's works. It closely resembles Eckert's V and VI, and also some of Wagner's projections.

This projection's mathematical derivation is presented in Uma Projeção Cartográfica Equivalente, Portuguese gzipped PostScript 92KB

The simple application I wrote to draw maps is somewhat restricted but effective. (luckily nowadays it takes seconds, not hours).

What Can Be Learned Here
The next pages present basic projection concepts, which properties are important for each map application, why there is a diversity of maps, how projections are designed, how maps can be misleading and how to choose a good projection for world maps.
More than a catalog of projections, I have attempted to present cartographical concepts in context; unavoidably, several important
projections are discussed in more than one place (e.g., while explaining its mathematical foundations and when listing projections with similar features).

**Basic Definitions and Concepts**

**The Shape of the Earth**

Since a map is a representation, the original shape of the represented subject must first be defined. An important branch of cartography, **geodesy** studies the Earth shape and how it is related to its surface's features.

**Spheres, spheroids and geoids**

**Geoid**, from the Greek for "Earth-shaped", is the common definition of our world's shape. This recursive description is necessary because no simple geometric shape matches the Earth:

- like in all space bodies above a certain mass, the Earth's materials aggregated in a spherical shape, which minimizes gravity and potential energy
- however, quick rotation around its axis caused a bulging at the middle (Equator) and a flattening at the poles; the resulting shape is called an **spheroid** or **oblate ellipsoid**. The equatorial diameter is nearly 1/300 longer than polar diameter
- on average, the surface rests perpendicular to the gravitational force at each point, which also influences land level. However, mass concentration is not uniform, due to irregular crust density and land distribution. In addition to the rotation bulge, some researchers concluded that the southern hemisphere is expanded and its pole depressed, while the other half is compressed with a raised pole (the resulting shape resembles a pear, but average distortion of curvature does not exceed 50 meters). Others have the opinion that the Equator itself is elliptical
- finally, the surface is not smooth, further complicating the shape

Taken in account, those factors greatly complicate the cartographer's job but, depending on the task, some irregularities can be ignored. For instance, although important locally, terrain levels are minuscule in planetary scale: the tallest land peak stands less than 9km above sea level, or nearly 1/1440 of Earth diameter; the depth of the most profound sea abyss is roughly 1/1150 diameter.

For maps covering very large areas, especially worldwide, the Earth may be assumed perfectly spherical, since any shape imprecision is dwarfed by unavoidable errors in data and media resolution. This assumption holds
for most of this document. Conversely, for very small areas terrain features dominate and measurements can be based on a flat Earth.

**The Datum**

For highly precise maps of smaller regions, the basic ellipsoidal shape can not be ignored. A geodetic **datum** is a set of parameters (including axis lengths and offset from true center of the Earth) defining a reference ellipsoid. For each mapped region, a different datum can be carefully chosen so that it best matches average sea level, therefore terrain features. Thus, data acquisition for a map involves **surveying**, or measuring heights and distances of reference points as deviations from a specific datum (a delicate task: due to mentioned irregularities, gravity - and therefore plumb bobs and levers - is not always aligned towards the center of the Earth).

Several standard datums were adopted for regional or national maps. International datums do exist, but can not fit any particular area as well as a local one.

**Coordinate Systems**

**Latitudes and Longitudes**

Although the Earth is a three-dimensional object, when supposed spherical its surface has a constant radius, so any point on it is uniquely identified using a polar two-coordinate system.

In this wooden sphere (with an octant removed for clarity), the copper arrows define the coordinate system's origins. One arrow lies over the north-south geographical axis, while the other is defined by convention. The white surface "point" is located using two angles or coordinates, called the **longitude** (\( \lambda \)) and

A **graticule** is a spherical grid of coordinate lines over the planetary surface, comprising circles on planes normal (perpendicular) to the north-south axis, called **parallels** (red) and semicircular arcs with that axis as chord, called **meridians** (blue). True to their name, no parallels ever cross one another, while all meridians meet at each geographic
**latitude** ($\phi$). Every point has a single counterpart at maximum distance on the opposite side, called its **antipode** (not shown here).

Both sets of parallels and meridians are infinite, but of course only a subset is included in any map. A point's latitude and longitude (usually measured in degrees) define the crossing of a parallel and a meridian, respectively. So, latitudes mean north-to-south angles from a reference parallel, while longitudes mean west-to-east angles from a reference meridian.

**Parallels and Their Properties**

A natural reference, the longest parallel divides the Earth in two equal **hemispheres**, north and south; thus its name, **Equator**. Four other important parallels are defined by astronomical constraints. The geographical north-south axis is actually tilted slightly less than 23.5° from the plane of the Earth's orbit around the sun. This accounts for the different seasons and different lengths of day and night periods throughout the year.

![Schematic cross section of Earth's orbit. AT is the axial tilt, about 23.5°](image)

Every year about December 21st, the solar rays fall vertically upon a parallel near 23.5°S. That is the longest day in the southern hemisphere (notice how most of it is exposed to the sun, so that date is called the southern **summer solstice**), but the shortest day in the northern hemisphere (therefore **winter solstice**); not only shorter daylight periods but a shallower angle of incidence of solar rays explain the lower temperatures north of Equator.

Near June 21st, a similar phenomenon happens along the parallel opposite North. By definition, these two parallels encircle the torrid or tropical zone; they are named after the zodiacal constellations where the sun is at those dates, thus **Tropic of Capricorn** (south) and **Tropic of Cancer** (north). In regions south of the Tropic of Capricorn the sun appears to run always north of the observer, even at noon; in places north of the Tropic of Cancer the sun runs always south of the observer, while in tropical
regions the sun appears sometimes south, sometimes north, depending on the season.

Subtracting the axial tilt from 90° we get the latitudes of the Arctic (about 66.5°N) and Antarctic (about 66.5°S) polar circles. At December 21 the sun does not set at the Antarctic circle for a full day. Going south, we get even longer consecutive daylight periods, up to six months at the pole. There are correspondingly long nights at the Antarctic winter. Of course, the same occurs at the northern latitudes, with a six-month offset.

Points on the same parallel suffer similar rates of exposure to the sun, therefore are prone to similar climates (disregarding other factors like altitude, wind/sea conditions and terrain).

A point's latitude can be inferred from the sun's angle above the horizon at noon (the moment when the sun appears highest at the sky and a pole projects its shortest shadow). Sailors use instruments like the sextant for that job.

![East-west distances between points separated by one minute of solar time at different latitudes. Distance is zero at poles, where one "sees" every moment of the day at any time.](image)

**Meridians and Their Properties**

All points on a meridian have the same solar, or local, time. Due to different day lengths throughout the year, correction formulas are applied to convert it to a local mean time. Since it would be impractical having nearby regions with different time reckonings (one nautical mile, approximately 1853 meters, corresponds to an angle of 0°1" along the Equator, or a temporal difference of 4 seconds), the world is divided in 24 fuses, or time zones, each 15° wide. For everyday purposes, every point inside a zone is considered having the same standard time (actually, a few countries still use solar time). In practice, the time-jumping boundaries seldom follow the meridians, bending (usually at national or regional borderlines) to keep related places conveniently synchronized.

Unlike the Equator, there's no easily defined prime or "main" meridian, which was fixed (mainly by political consensus) in 1884 over the Royal
Observatory in **Greenwich**, near London, UK. This choice's only obvious advantage is setting the opposite meridian (near or at the left or right edges of many world maps) away from most inhabited areas. That opposite meridian is the base of the **international date line**, which separates world halves in two different days. Again, this line is somewhat irregular in order to keep national territories (mostly Pacific islands) in a single timezone. Compared to finding a point's latitude, getting its longitude is a much more involved procedure, usually comparing the time separating the noon at the reference meridian and at the point in question.

**Maps, Globes and Projections**

Any study in geography requires a reduced model of the Earth, like a globe or map. Neither is perfect: a globe is seldom [practical](https://en.wikipedia.org/wiki/Globe), and flat maps are never free from [errors](https://en.wikipedia.org/wiki/Map_error). Selecting or creating a good map involves interesting choices and trade-offs.

**What's a Projection?**

A cartographical **map projection** is a formal process which converts (mathematically speaking, maps) features between a spherical or ellipsoidal surface and a **projection surface**, often flat. Although many projections have been designed, just a few are currently in widespread use. Some were once historically important but were superseded by better options, some are useful only in very specialized contexts, while others are little more than fanciful curiosities.

**Projection Surfaces**

The map's support, the projection surface is usually created, i.e., [developed](https://en.wikipedia.org/wiki/Map_projection), conceptually touching the mapped sphere in one (surface is tangent) or more (surface is secant) regions. Intuitively, portions near the touching regions depart less from the original spherical shell and are more faithfully reproduced. Some projections are actually composites, fitting separate surfaces to different regions of the map: overall error is reduced at the cost of greater complexity.

Sometimes a conceptual auxiliary surface like a cone, open cylinder, ellipsoid or torus is employed: sphere features are transferred to that surface (many times by the laws of perspective), which is then flattened. Many projections are classified as "cylindrical" or "conic"; however, for most of them, the naming is just an analogy or didactic device, since they aren't actually developed on an intermediary surface; rather, the **resulting map** can be rolled onto a tube or a cone.

**The Unavoidable Distortion**
Two quasi-orthographic views of a sphere (plus polar axis) divided in eight equal sections, surrounded by four maps at the same scale using, clockwise from top right: azimuthal equidistant, Lambert's equal-area cylindrical, Maurer's equal-area star with four lobes, Winkel Tripel projections.

A few selected projections illustrate how the same spherical data can be stretched, compressed, twisted and otherwise distorted in different ways. The azimuthal equidistant map has interesting properties regarding directions and distance from the central pole, but the outer hemisphere is greatly stretched: its pole becomes a circle. Both poles become lines in the equal-area cylindrical map, but it covers the same area as the original sphere; also, all octants have identical shape. This particular star projection has unequal octants and marked loss of continuity; however, it also preserves area. In the Winkel Tripel map, octants have different shapes, area is changed and poles are linear, but overall distortion is subjectively smaller. Finally, the orthographic views, projections themselves, show only part of the sphere.

All projections suffer from some distortion; none is "best" for all purposes. Octants would assume even stranger shapes in oblique aspects.

No matter how sophisticated the projection process, the original surface's features can never be perfectly converted to a flat map: distortion, great or small, is always present in at least one region of planar maps of a sphere. Distortion is a false presentation of angles, shapes, distances and areas, in any degree or combination.

Every map projection has a characteristic distortion pattern. An important part of the cartographic process is understanding distortion and choosing the best combination of projection, mapped area and coordinate origin minimizing it for each job.
Cones and cylinders are **developable** surfaces with zero Gaussian curvature (in a nutshell, at every point passes a straight line wholly contained in the surface). Distortion always occur when mapping a sphere onto a cone or cylinder, but their reprojection onto a plane incurs in no further errors.

**The Choice of Coordinate Origin**

Another key feature of any map is the orientation (relative to the sphere) of the conceptual projection surface.

A particular projection may be employed in several **aspects**, roughly defined by the graticule lay-out and the sphere's region nearest the projection surface, commonly the center of a whole-world map:

- a **polar** map aligns the Earth axis with the projection system's, thus one of its poles lies at the map's conceptual center;
- an **equatorial** map is centered on the Equator, which is set across one of the map's major axes (mostly horizontally);
- an **oblique** map has neither the polar axis nor the equatorial plane aligned with the projection system.

Also,

- the most "natural" aspect of a projection, called **normal**, **conventional**, **direct** or **regular**, is ordinarily determined by geometric constraints; it often demands the simplest calculation and produces the most straightforward graticule. The polar aspect is the normal one for the **azimuthal** and **conic** groups of projections, while the equatorial is the normal for **cylindrical** and **pseudocylindrical** groups.
- the **transverse** aspect frequently resembles the normal one, except by a simple rotation of 90°.
Three (normal, transverse and oblique) aspects applied to three (azimuthal equal-area, Gall’s stereographic cylindrical and Albers’s conic) projections with different tangent projection surfaces in blue (just a few of infinitely many possible oblique maps are presented). Some projections may actually be derived via perspective geometry; for most, however, surfaces are only illustrative: the map may be laid on a developable surface, but is not calculated from it.

Distinctive graticules of some projection groups (radially symmetric meridians in azimuthal and conic maps, rectangular grid in cylindrical maps) are only realized using their simple, normal aspects. Trivial rotations of the finished map, like turning it sideways or upside-down, leave both aspect and projection unchanged. On the other hand, modifying the aspect does not affect either represented area or the shape of the whole map.

Theoretically (especially supposing a spherical Earth) any projection may be applied in any aspect: after all, the parallel/meridian system is a convention which might have origins anywhere, although it is hard imagining others more useful than the poles. However, many projections are almost always used in particular aspects:

- their properties may be less useful otherwise. E.g., many factors like temperature, disease prevalence and biodiversity depend on climate, thus roughly on latitude; for projections with constant parallel spacing, on equatorial aspects latitude is directly converted to vertical distance, easing comparisons.
- several projections whose graticules in normal aspects comprise simple curves, like straight lines and arcs of circle, were originally defined by geometric construction. Since many non-normal aspects involve complex curves, they were not systematically feasible before the computer age (indeed, mapping was an important motivation for calculation shortcuts like logarithms).
Even though **oblique aspects** are frequently useful, in general calculations for the actual ellipsoid are fairly complicated and are not developed for every projection.

**Useful Map Properties**

Matching Projection to Job

No map projection is perfect for every task. One must carefully weigh pros and cons and how they affect the intended map's purpose before choosing its projection. The next sections outline desirable properties in a map, mentioning how projections can be used or misused.

Like in the real Earth, this globe's north-south axis is tilted at 23°27'30" from the orbital plane.

For any map, the most important parameters of accuracy can be expressed as:

- can **distances** be accurately measured?
- how easy is obtaining the **shortest path** between two points?
- are **directions** preserved?
- are **shapes** preserved?
- are **area ratios** preserved?
- which regions suffer the most, and which kind of, **distortion**?
Globe Properties

Unfortunately, only a **globe** offers such properties for *any* points and regions. Since crafting a globe is only a matter of reducing dimension (no projection is involved), every surface feature can be reproduced with precision limited only by practical size, with no loss of shape or distance ratios. As a bonus, a globe is a truly three-dimensional body whose surface can be embossed in order to present major terrain features. But globes suffer from many disadvantages, being:

- bulky and fragile, clumsy to transport and store;
- expensive to produce, especially at larger sizes, thus impractical for showing fine details;
- difficult to look straight at every point, therefore
- cumbersome for taking measurements or setting directions;
- able to show a single hemisphere at a time;
- completely unfeasible for widespread reproduction by printed or electronic media

Exploiting Map Properties

So, flat map projections are usually more important despite their shortcomings. In particular, no flat map can be simultaneously **conformal** (shape-preserving) and **equal-area** (area-preserving) in every point.

However, a reasonably small spherical patch can be approximated by a flat sheet with acceptable distortion. In most projections, at least one specific region (usually the center of the map) suffers little or no distortion. If the represented region is small enough (and if necessary suitably translated in an **oblique** map), the projection choice may be of little importance.

On the other hand, the fact that no projection can faithfully portray the whole Earth should not lead to a pessimistic view, since distorting the planet on purpose makes possible - unlike a globe - uncovering important facts and presenting at a glance relationships normally obscure. Skillfully used, distortion is a powerful visual tool; this becomes explicit in a kind of pseudoprojection called **cartogram**, where the place a point is drawn depends not only on its location on Earth, but also on attributes of the mapped region (like a county's population or a country's economic output).

No projection is intrinsically good or bad, and a projection suitable for a particular problem might well be useless or misleading if applied elsewhere.
Identically marked graticules in orthographic (representing the original globe) and van der Grinten III projections at same scale.

The projection known as Van der Grinten's third violates all five graticule properties:

1. purple lines are very stretched near the poles; green lines are longer farther from the vertical axis (the constant vertical spacing between parallels is deceptive)
2. red lines are also slightly stretched closer to the map edges
3. the blue cells are also enlarged near the edges; this is more obvious at high latitudes
4. of all meridians, only the central remains straight
5. parallels and meridians cross at right angles only at the Equator and central meridian; elsewhere, there's shearing, symmetrical around the vertical axis: angles are compressed in opposite directions east and west of the central meridian

The Graticule as a Guide to Distortion

Especially for a map in the normal aspect, a quick visual inspection of its graticule provides obvious clues of whether its projection preserves features. For instance, if the coordinate grid is uniformly laid (say, one line every ten degrees),

1. along any meridian, the distance on the map between parallels should be constant
2. along any single parallel, the distance on the map between meridians should be constant; for different parallels, should decrease to zero towards the poles
3. therefore, any two grid "cells" bounded by the same two parallels should enclose the same area

Also,
4. the Equator and all meridians should be straight lines, since they don't change direction on the Earth's surface
5. any meridian should cross all parallels at right angles

Again, for any particular projection, violation of any or all these properties doesn't necessarily make it poorly designed or useless; rather, it suggests (and constrains) both the range of applications for it is suitable and, for each application, regions on the map where distortion has significance.

**Can Distances be Accurately Measured?**

![Equidistant cylindrical map](image)

In this [equidistant cylindrical] map, all arrow lines have the same length; however, actual world distances (measured in km) between the ends of each arrow vary enormously. The included graphical scale (blue) is only useful along the Equator and meridians (where scale is constant, for this projection at least); also, only along those lines each arrow actually follows a straight route on Earth.

Any practical map shrinks Earth features down to a manageable size. The general rate of reduction is called a **scale**, and frequently a small "ruler" superimposed on the map eases distance evaluation. For instance, on a 1:100000-scale map, two points separated by 1 unit could represent two cities actually 100000 units apart. With a scale 10 times "larger" - 1:10000 - the same two cities would appear separated by 10 units on the map (therefore, if the same region is presented in different maps, smaller-scale versions can be more portable but probably less precise and detailed).

However, only a true **globe** would allow a similar conclusion for any pair of points on its surface. In flat maps, most likely the scale will *not* be constant, changing with direction and location. As a result, although useful for rough estimation, scaling rulers and numeric ratios are misleading except in very large scales or, conversely, small mapped areas (more elaborate small-scale maps have a set of rulers and guides for their use).
On the map on the left, the center C of the blue cross looks 41% farther from the tips (like B) of the red line than from its center D. Actually, every point on the line is equally far from the cross. Distance between A and B is actually zero, since both lie on a pole. The map on the right puts those facts in a better perspective, in more than one sense.

Actually, there is another reason why distances across a map can deceive the unwary reader: the shortest distance between two points on a sphere is rarely represented by a straight line on a map.

Lines (straight or not) in the map with constant scale and length proportional to corresponding lines on Earth are called standard lines. Map projections with a well-defined, nontrivial set of standard lines are sometimes called equidistant.

In a Sanson-Flamsteed equatorial map, all parallels are standard lines: if segment A is twice the length of B, corresponding lines on Earth follow the same proportion. Although the straight vertical distances between parallels follow this rule, distances along meridians do not, except for the central one.

In this cylindrical equidistant map, only vertical lines and the Equator preserve a constant scale. Since all parallels are equally long in the map, horizontal scale increases quickly towards the top and bottom, reaching infinity at the poles.

An azimuthal equidistant map preserves distances along any lines through the central point. This important projection must be custom-made for every location of interest.

The cordiform Werner projection also has standard lines radiating from the center. In addition, every arc centered in that point is also a standard line.
Useful Map Properties: the Geodesic

What is the Shortest Path Between Two Points?

On any spherical surface, the shortest path between any two points is part of a *geodesic line* (also called *geodetic, orthodrome* or *great circle path*) passing through the points and centered on the sphere. Except due to constraints like traffic patterns and weather conditions, long-distance travellers like aircraft pilots always seek for the shortest route. For that purpose, a map showing great circles as straight lines is better suited. Unfortunately, no map projection can show true geodesics between *any* pair of points as such. The following examples show in red a great circle connecting Campinas, Brazil, and Tokyo, Japan.

The elliptical *projection created by Mollweide* preserves area ratios but not directions. Regions near the top and bottom suffer from greater distortion the farther the distance from the map center. Notice how map coordinates are translated prior to projection in order to move the region of interest to a lesser-distortion area and to emphasize the circular (on Earth) red path.

The *equidistant cylindrical* projection does not preserve either areas or directions. Only distances are preserved along the meridians and the Equator. Sadly, most cylindrical projections are probably chosen due to their neat rectangular lay-out rather than any outstanding cartographic property.
The **azimuthal family** of projections shows true directions from the center point (*azimuth*) only. The **azimuthal stereographic** projection is *circle-preserving*, as any circle upon the sphere (every geodesic, parallel and meridian) is still mapped to a circle; in particular, all geodesic lines crossing the central point map to circles with infinite diameter, i.e. straight lines. Unfortunately, it can show only one hemisphere at a time. Another very important azimuthal projection, the **gnomonic**, maps into straight lines all great circles, even those not passing through the central point, but can present even less than one hemisphere. The **azimuthal equidistant** can include the whole world and presents true direction and distance to any point from the center while suffering from lesser distortion near the map periphery.

Projection distortion and unfamiliar shapes could make difficult realizing the red lines in previous maps as actual circles. The **azimuthal orthographic** projection clearly shows the Earth's sphericity as seen from a vantage point far away in space. This closely mirrors the actual expedient of finding a geodesic by applying a taut line against a globe.

**Useful Map Properties: Directions**

**Are Directions Preserved?**

A compass is the instrument of choice for following a prescribed route. It shows the deviation from a standard direction, called a *bearing* or *course* (for our purposes, magnetic declination is irrelevant).
1. Problem: suppose one leaves home by plane, keeping a constant course bearing. Which localities will be visited?
2. Problem: given two points on a map, which bearing must be followed to travel between them?

The two problems are related and can not be easily solved for every location in most maps, as general directions are seldom preserved. In a (theoretically) perfect map, meridians and parallels must cross at right angles in every point but the poles.

A loxodrome (or rhumb line) is a line of constant bearing. It is the easiest route between two points since a constant bearing is enough to follow; any other path would require frequent changes of direction. Loxodromes are an invention of Pedro Nunes (ca. 1533), after suggestions by Martim Afonso de Sousa.

The blue line shows the loxodrome as a path starting near Campinas, Brazil, with constant bearing 60° clockwise from true North. The North Pole is reached after an infinite number of tighter and tighter turns. The Mollweide map is equal-area but suffers from strong shape distortion near the poles. Reading the loxodrome is a bit easier in the cylindrical equidistant projection.

The orthographic projection helps viewing the rhumb line's constant angle with every parallel and meridian. In a stereographic map, the rhumb line maps to a logarithmic spiral, the plane curve which intersects every radius at a constant angle and looks the same no matter how magnified.

Actually, any rhumb line is part of a curve which winds from pole to pole, called a spherical helix.
Mercator's most famous projection is unique: every loxodrome is drawn as a straight line, making trivial finding the bearing between any two points. However, the Mercator map alone is not enough for general navigation. Also, in this equatorial form, the polar regions can not be included (here the loxodrome has constant slope and infinite length; therefore the North Pole should be infinitely far up).

Useful Map Properties: Shapes

Are Shapes Preserved?
A conformal (or orthomorphic) map locally preserves angles. Thus, any two lines in the map follow the same angle as the corresponding original lines on the Earth; in particular, projected graticule lines always cross at right angles (a necessary but not sufficient condition). Also, at any particular point scale is the same in all directions. It does not follow that shapes are always preserved across the map, as any conformal map includes a scaling distortion somewhere (that is, scale is not the same everywhere). Any azimuthal stereographic or Mercator maps are conformal.

Loxodromes and geodesics

A straight line drawn on a Mercator map connecting Campinas, Brazil, to Seoul, South Korea is a loxodrome at a constant angle of approximately 79°39' from any meridian. An aircraft taking off from Campinas would easily land in Seoul following this fixed bearing (disregarding factors like traffic airlines, wind deviation, weather, national airspaces and fuel range; actual customary routes go westwards but are in fact similar) along the whole trip.

However, that easy route would not be the most economical choice in terms of distance, as the geodesic line shows.

The two paths almost coincide only in brief routes. Although the rhumb line is much shorter on the Mercator map, an azimuthal equidistant map tells a different story, even though the geodesic does not map to a straight line since it does not intercept the projection center.

Since there is a trade-off:
following the geodesic would imply constant changes of direction (those are changes from the current compass bearing and are only apparent, of course: on the sphere, the trajectory is as straight as it can be)
following the rhumb line would waste time and fuel,

a navigator could follow a hybrid procedure:

1. trace the geodesic on an azimuthal equidistant or gnomonic map
2. break the geodesic in segments
3. plot each segment onto a Mercator map
4. use a protractor and read the bearings for each segment
5. navigate each segment separately following its corresponding constant bearing.

Useful Map Properties: Areas

Are Area Ratios Preserved?
Apart from navigational purposes, for most maps of general interest this is the most essential property. Since a map is an instrument to convey information at a glance (much faster and more concisely than a table of numbers), it is important to properly display true area ratios. Especially important applications include:

- scientific divulgence of geographical distributions like welfare levels, pollution/deforestation, crop yieldings, greenhouse warming and the like
- educational atlases and charts, and press information
The latter issue is quite worrisome as school and college books (to say nothing of newspapers and TV shows) are usually very careless in their choice of map projections. This is unfortunate, as maps make powerful and long-lasting visual images; a wrongly presented map could permanently distort one's view of the world (however, this very argument was also used as misleading and myopical propaganda).

As a classical example, Mercator's projection (an extremely powerful tool in the proper context) is frequently - and sadly - used for popular illustration, where its conformality is useless and areal ratios misleading.

In the conventional aspect of Mercator's projection, surface stretching increases quickly but continuously to infinity towards the poles. Even Greenland's northern half looks much broader than the southern one - this projection preserves shapes but only locally. In contrast, Mollweide's elliptical projection correctly shows the correct size proportion between map features.

An equal-area (also called equivalent or authalic) projection preserves areal relationships; in other words, given any two regions A and B on the Earth and the corresponding regions A' and B' on an equal-area map, the surface ratios A/A' and B/B' are identical (A and B need not have the same shape; shapes A and A' will probably be different).
Although Mercator's projection makes Africa (one of the largest continents, 29,800,000 km²) and Greenland (the largest island, 2,175,600 km²) apparently similar in size, an area-preserving projection shows their true area ratio - about 13.7 : 1 - much more clearly.

Useful Map Properties: Distortion Pattern

Assessing and Measuring Distortion
Every flat map includes some distortion of shape, area or length. Some regions might be free of distortion while others could suffer from severe error. Assessing the most affected regions is useful in choosing an appropriate projection.

Tissot's Indicatrix
An important tool introduced by Nicolas Tissot in the 19th century is known today as Tissot's indicatrix. Suppose a small circle drawn upon the original sphere. When mapped to a flat surface, the circle could:

- preserve its shape and size, thus being free of distortion
- get smaller or larger, thus suffering a scale distortion
- suffer from a shape deformation

Several very small circles set along the sphere and projected onto a map show the latter's distortion pattern. Hammer's elliptical projection is not conformal and deforms the Tissot circles, except at the very center of the map. Shape distortion is also obvious in meridians, which are "straight" over the terrestrial surface, but curved lines here. However, since it is an equal-area projection, all indicatrices cover exactly the same area.

On the other hand, the rectangular Mercator projection is conformal: all indicatrices remain circular in shape, parallels keep parallelism, meridians are straight lines and always perpendicular to every parallel. Areas are not preserved, but greatly increase towards the top and bottom of the map: circles at the poles would be infinitely large (this is to be expected,
since meridians cross one another on a sphere but never touch in a Mercator map. Only infinite circles on different meridians could be all concentric as in the globe's poles).

Equatorial Mercator map, clipped at 85°N and 85°S, with identical indicatrices (theoretically infinitesimally small; the greatly oversized circles here used for illustration slightly violate conformality)

In a projection neither conformal nor equal-area like the azimuthal orthographic, Tissot indicatrices keep neither original shape nor area.

Ideally, for every circle centered at a meridian-parallel intersection, scale should be preserved in both directions, while the intersection angle should be 90°. Tissot developed a formula defining the angular deformation at any given point from the scales and angle distortion. The maximum angular deformation can be plotted on a map thereby presenting areas of major distortion.
Equatorial azimuthal orthographic map

As shown by the **oblique** orthographic and Mercator maps, Tissot indicatrices present an *overall* deformation pattern, not affected by graticule rotation (in the first case, distortion depends only on radial distance from center; in the latter, only on vertical coordinate). These are exactly the same patterns presented by the previous equatorial versions.

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**Oblique azimuthal orthographic map**

**Oblique Mercator**

**Equatorial cylindrical equidistant map**

Finally, the equatorial **equidistant cylindrical** map proves:

- at least the horizontal scale is always exaggerated in polar caps in every cylindrical map
- having parallels and meridians crossing at right angles everywhere is not enough for achieving conformality

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**How Projections Work**

What lies behind a projection? Which rules tell the cartographer where coastlines are plotted? How do we set a **mapping** (mathematically, a conversion between two sets of values) from sphere coordinates to planar points? How good is this transformation?
"Projective"/Perspective/Geometric vs. "Algorithmic" Projections

![Conceptual model of a pure perspective projection. Light rays pierce the Earth drawing a map on a flat square suspended in space. Notice how points correspond on Earth and map.](image)

Some projections of the **azimuthal**, **cylindrical** and **conic** families have a direct geometric interpretation as light rays **projected** from a source intercept the Earth and, according to laws of **perspective**, "draw" its features on a surface. The latter may be a plane, yielding the map itself, or an **intermediate shape** like a cylindrical or conical shell.

On the other hand, many projections are only distantly inspired by geometric principles. For instance, **Mercator's cylindrical projection** can't be visualized as a perspective process unless:

- light rays don't follow straight trajectories, or
- the light source is not a point or straight line, or
- the projection surface is not a simple tube

In all three cases the complexity negates the usefulness of a perspective model.

Indeed, many projections have simply no geometric or physical interpretation, and are described purely by mathematical formulae. I.e., the cartographer devises a spherical-to-flat mapping according to some desirable but arbitrary property or constraint.

**Forward and Inverse Formulas**

Perspective or not, a projection can be defined by two sets of mapping equations:

- **forward** or direct relation converts **polar** coordinates (longitude $\lambda$, latitude $\varphi$, Earth radius $R$) to **Cartesian** coordinates (horizontal distance $x$ from origin, vertical distance $y$), provided a convenient scaling factor (not to be confused with the map **scale**). Equations included here assume a unitary scaling factor.
- **inverse** relation performs the opposite transformation
Coordinate transformations defined by mapping relations

Usually those relations are not functions (e.g., the same point on the sphere may be represented by several points on the map). Instead of Cartesian distances, plane polar coordinates (radius $\rho$, angle $\theta$) can be used, being in fact easier to express for many projections.

Although not generally presented here, inverse mapping makes possible calculating location given a point on a map or an aerial/satellite photograph. Thus, it is relevant to several problems, like interactive mapping applications. It is of course important for converting an already projected map to other projections.

**Deriving Projection Formulae**

Even those without a mathematical background could get a fresh insight on the geographical sciences by understanding a projection formula or two; however, the reader can instead skip ahead to the main projection groups.

The next sections sketch the actual process for deriving mapping formulae for a few projections:

- the **azimuthal orthographic** projection, purely geometric, can be understood by anyone; basic trigonometry is involved only for algorithm derivation
- the **Braun stereographic cylindrical**, an arbitrary geometric projection
- the **Sanson-Flamsteed**, also called sinusoidal, a very plain algorithmic projection demanding only simple trigonometry
- the **Mollweide** or Babinet, a slightly more difficult projection solved by integral calculus and numerical analysis
- two generalized azimuthal projections, the **equidistant and equal-area**, in both polar and equatorial aspects
- the **equidistant cylindrical**, a very simple arbitrary projection, and two hybrid derivatives, **Winkel I** (generalized Eckert $V$) and II
- the **Aitoff, Hammer and Winkel Tripel** projections, derivatives from azimuthal and cylindrical maps

The map shape is defined beforehand for all but the Sanson-Flamsteed and Winkel projections, where it is a consequence of the projections's
constraints. Only the Mollweide, azimuthal equal-area and Winkel derivations require basic calculus, numerical methods, or both.

Instead of commonplace degrees, minutes and seconds, in cartographic mathematics all angles of latitude and longitude are more usefully measured in radians, since the length of a circular arc can be directly calculated by its radius multiplied by the angle in radians. E.g., a straight angle of 180° is equivalent to \( \pi \) radians; all points at latitudes 60°N and 60°S are \( R\, \pi / 3 \) units away from the Equator. Northern latitudes and eastern longitudes are arbitrarily considered positive angles; e.g., 45°S is expressed as \(-\pi / 4\).

**Deducing the Azimuthal Orthographic Projection**

The purely geometrical azimuthal orthographic projection can be entirely visualized as a physical model. The easiest case, a polar aspect, is presented here for the Northern hemisphere.

Suppose the Earth lying over a plane parallel to the Equator. Light rays emanating from a point infinitely far away on the north-south polar axis pierce a semitransparent northern hemisphere and "draw" its features onto the plane. The southern hemisphere is considered completely transparent.

Alternatively, one can imagine an observer infinitely far away on that axis. Parallel light rays emanating from Earth’s surface hit an intervening plane (or disc) perpendicular to the rays, where the image is developed. Since all rays are parallel ("cylindric perspective"), the plane may or may not be tangent to the sphere without affecting the result.

Anyway, only one hemisphere can be seen at any time.

Geometrically, the orthographic projection can be imagined as converting polar coordinates to 3-D cartesian, then flattening it (ignoring one coordinate). The projection formulae are easily derived in polar coordinates:

\[
r = R \cos \varphi, \quad 0 \leq \varphi \leq \pi / 2, \quad \theta = \lambda, \quad \rho = r\]
On the left, Earth set for a polar azimuthal orthographic projection, resting on the projection plane; distance of projected point to center of map depends only on latitude. On the right, Earth and projection plane viewed from above; angle of point around center of map equals longitude.

Only points with $\varphi \geq 0$ (for the north polar case) or $\varphi \leq 0$ (south polar) are visible.

Converting to Cartesian coordinates,

\[ x = \rho \cos \theta = R \cos \varphi \cos \lambda \]
\[ y = \rho \sin \theta = R \cos \varphi \sin \lambda \]

Conversion of the equations to inverse mapping is straightforward.

A more general aspect, either equatorial or oblique, can be obtained by first rotating Earth coordinates in 3D space, then applying the previous equations. Early cartographers designed general orthographic maps by first plotting the polar aspect, then locating key grid points by a set of parallel lines drawn from the polar map.

Two other important azimuthal projections are created just by changing the light source's position.
For a practical presentation, a cartographer could conceivably paint coastlines and other geographical features onto a glass globe or bowl and, using reflected sunlight or a strong flashlight at a convenient distance, project the globe shadows on the wall, thus creating a variety of azimuthal projections like the orthographic and stereographic.

Deducing Braun's Stereographic Cylindrical Projection

The stereographic cylindrical projection designed by Braun can be visualized geometrically, but it is slightly more complicated than the previous azimuthal orthographic example. Instead of a flat projection plane directly yielding the projected map, here the projection surface is a cylindrical sheet tightly rolled against the Equator. Every meridian is drawn on this tube by light rays emanating from an equatorial point on the meridian directly opposite. The tube is then cut along an arbitrary meridian and unrolled.

This is a general procedure for creating some cylindrical projections, although not all of them employ such a simple model. In fact, projections like the equidistant cylindrical are defined by arbitrary constraints, not a perspective process.
Transversal cut of the tube tangent at Equator. Due to projection constraints, the length of tube (therefore the map's height) is twice the sphere's diameter.

Equations for direct mapping are straightforward.

Projected meridians are straight vertical lines, regularly spaced, thus \( x = R \lambda \).

Parallel \( \phi \) is projected as a straight horizontal line, whose \( y \)-coordinate can be derived from the diagram: by a simple proportion, \( h / (w + R) = y / (R + R) \). Thus,

- \( h = R \sin \phi \)
- \( w = R \cos \phi \)
- \( y = 2R \sin \phi / (1 + \cos \phi) \)

Making \( \theta = \phi / 2 \) and using the trigonometric identities \( \sin 2\theta = 2 \sin \theta \cos \theta \) and \( \cos 2\theta = 1 - 2 \sin^2 \theta \), \( y = 2R \sin \theta \cos \theta / (1 + 1 - 2 \sin^2 \theta) = 4R \sin \theta \cos \theta / 2 (1 - \sin^2 \theta) = 2R \sin \theta \cos \theta / \cos^2 \theta \). Therefore,

- \( x = \lambda R \)
- \( y = 2R \tan (\phi / 2) \)
Again, inverse mapping equations can be easily obtained.

Deducing the Sanson-Flamsteed Projection

Given the Earth radius $R$, suppose the equatorial aspect of an equal-area projection with the following properties:

1. Parallels map into equally spaced parallel straight lines
2. All parallels are standard lines
3. The central meridian is a standard line

Such a map would be appropriate for certain meteorological presentations since linear parallel spacing make easy comparing latitude effects; constant scale on parallels help measuring main wind speeds. Finally, areas are preserved so comparing choropleths and other color-coded datasets makes sense.

Imagine at first an Earth-sized map; since $-\pi / 2 \leq \varphi \leq \pi / 2$, and parallels are uniformly spaced, $y$-coordinates are proportional to latitude only; $-\pi R \leq y \leq \pi R$, thus $y = \varphi R$. We want an area-preserving map, so the circumference of any parallel equals the Earth circumference at that latitude. The radius of a spherical cap at angle $\varphi$ is $R \cos \varphi$. Therefore, the corresponding projected parallel has length $2k\pi R \cos \varphi$. At the Equator $\varphi = 0$, parallel length is $2\pi R$, thus $k = 0.5$.

Since horizontal scale is constant and $-\pi \leq \lambda \leq \pi$, $x / \pi R \cos \varphi = \lambda / \pi$.

The resulting transformation

- $x = R\lambda \cos \varphi$
- $y = R\varphi$

yields the so-called Sanson-Flamsteed projection, also known as the sinusoidal for obvious reasons.
Mathematically one of the simplest projections, it has fairly satisfactory results except perhaps at higher latitudes. One could use oblique Sanson-Flamsteed maps for a clearer view of polar areas (at the cost of losing the parallel spacing property), or interrupted versions avoiding high-latitude shearing.

**Deducing Mollweide's Projection**

Although mathematically very simple, the previous Sanson-Flamsteed projection is not completely satisfactory at high latitudes, due to excessive shearing and crowded meridians. A slightly more complicated analysis leads to a new, somewhat complementary projection.

Given the Earth's radius $R$, suppose the equatorial aspect of an equal-area projection with the following properties:

1. A world map is bounded by an ellipse twice broader than tall
2. Parallels map into parallel straight lines with uniform scale; only the Equator is a standard line
3. The central meridian is a straight standard line; all other ones are semielliptical arcs

Suppose an Earth-sized map; let us define two regions, $S1$ on the map and $S2$ on the Earth, both bounded by the Equator and a parallel. The equal-area property can be used to calculate $y$ given $\phi$. Given $y$ and $\lambda$, $x$ can be calculated immediately from the ellipse equation, since horizontal
scale is constant.

Equation of ellipse centered on origin, major axis on x-axis:
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \] ; \[ x^2 = a^2(1 - \frac{y^2}{b^2}) \] ; for \( -b \leq y \leq b \), \( x = \frac{a}{b} \sqrt{b^2 - y^2} \)

Area between x-axis and parallel mapped into \( y = y_1 \):

\[ S_1 = 2 \int_0^{y_1} xdy = 2 \frac{a}{b} \int_0^{y_1} \sqrt{b^2 - y^2} dy \]

Let \( y = b \sin \theta, 0 \leq y \leq b, 0 \leq \theta \leq \pi/2 \), \( dy = b \cos \theta d\theta \)

\[ \int \sqrt{b^2 - y^2} dy = \int \sqrt{b^2(1 - \sin^2 \theta)} d\theta = b \int \cos \theta \cos \theta d\theta = b^2 \int \cos^2 \theta d\theta \]

Since \( \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} \), \( b^2 \int \cos^2 \theta d\theta = \frac{b^2}{2} (\int \cos \theta d\theta + \int \cos 2\theta d\theta) \)

Since \( \int \cos \alpha d\alpha = \frac{1}{n} \sin n\alpha + C \), \( \int \sqrt{b^2 - y^2} dy = \frac{b^2}{2} (\theta + \frac{\sin 2\theta}{2}) + C \)

\[ S_1 = \left. 2ab \frac{2\theta + \sin 2\theta}{2} \right|_{\theta = 0}^{\theta = \frac{\theta_1}{2}} + C \]

for some \( 0 \leq \theta_1 \leq \frac{\pi}{2} \)

Area of 2:1 ellipse is \( ab\pi = \frac{a^2\pi}{2} \)

Since area of Earth = \( 4\pi R^2 \), \( a^2 = 8R^2 \Rightarrow a = R\sqrt{8} \)

\[ S_1 = 2R^2(2\theta_1 + \sin 2\theta_1) \]

On a sphere, the area between the Equator and parallel \( \varphi \) is

\[ S_2 = 2\pi Rh = 2\pi R^2 \sin \varphi \]

\[ S_1 = S_2 \Rightarrow 2R^2(2\theta_1 + \sin 2\theta_1) = 2\pi R^2 \sin \varphi \]

\[ 2\theta_1 + \sin 2\theta_1 = \pi \sin \varphi \]

\( \theta_1 \) must be found by interpolation or successive approximations.

Finally, since horizontal scale is uniform,

\[ y = \frac{R}{2} \sqrt{8 \sin \theta} = \sqrt{2R \sin \theta_1} \]

\[ x = \frac{2a}{\pi} \sqrt{2R^2 - y^2} = 2\sqrt{2} \frac{R^2}{2R^2 - 2R^2 \sin^2 \theta_1} \frac{\lambda}{\pi} = 2\sqrt{2} \frac{R^2}{2R^2} \cos \theta_1 \]
Two Aspects for Two Arbitrary Azimuthal Projections

General Polar Azimuthal Projections

The mathematical development of the azimuthal orthographic projection is purely geometric. Even though some azimuthal projections do not follow such a perspective process, all can be reduced to a general pattern, given the tangency point $T$:

For the North polar aspect, $\theta = \lambda$.

The Azimuthal Equidistant Projection

For the azimuthal equidistant, an important projection for navigational applications, distance $\rho$ from the center of the map is directly proportional to radial distance from the tangent point. In the North polar aspect:

$$\rho = (\pi / 2 - \phi)R$$

The austral aspect is just as easy, with

$$\rho = (\pi / 2 + \phi)R$$ and $\theta = -\lambda$

Lambert's Azimuthal Equal-area Projection

In the only projection both azimuthal and equal-area, created by Lambert and suitable for world maps, distance from the center of the map is progressively reduced in order to keep areal equivalence. Formulas follow from basic integral calculus.
Area element on sphere, given colatitude $\Phi = \pi / 2 - \phi$:
$$ds = 2nR \sin \Phi R \, d\Phi = 2nR^2 \sin \Phi \, d\Phi$$

Corresponding element on map: $ds = 2nr \, dr$

For a given $\Phi_1$, we want $\rho$:
$$\int_0^{\phi_1} 2\pi R^2 \sin \phi \, d\phi = \int_0^\rho 2\pi r \, dr \iff 2\pi R^2 \int_0^{\phi_1} \sin \phi \, d\phi = 2\pi \int_0^\rho r \, dr$$

$$-R^2 \cos \phi \bigg|_0^{\phi_1} = \frac{r^2}{2} \bigg|_0^\rho \Rightarrow \rho^2 = -2R^2 (\cos \phi_1 - 1)$$

$$\rho = R\sqrt{2\sqrt{1 - \cos \phi_1}} = 2R\sqrt{\frac{1 - \cos \phi_1}{2}}$$

$$= 2R \sin \frac{\phi_1}{2} = 2R \sin \frac{\pi / 2 - \phi}{2}$$

A similar sign change applies if the south polar aspect is intended.

Combined azimuthal equidistant (top) and equal-area (bottom) polar maps

The combined map shows both projections virtually identical at latitudes above 70° N. Beyond that, parallels get closer and closer together in Lambert’s half, while remaining equally spaced in the azimuthal equidistant portion. The resulting areal difference is clearly visible in Antarctica.
General Equatorial Aspect for Azimuthal Maps

Calculating other aspects for azimuthal maps is possible applying coordinate transformations and rotations in space. However, the important equatorial aspect can be obtained in a more direct way, using two properties of triangles on a spherical surface.

Given $A$, $B$, $C$ angles on a spherical triangle's vertices, and $\alpha$, $\beta$, $\gamma$ the corresponding angles between edges connecting triangle vertices and center $O$ of sphere,

Law of sines: \[
\frac{\sin A}{\alpha} = \frac{\sin B}{\beta} = \frac{\sin C}{\gamma}
\]

Law of cosines: \[
\cos \gamma = \cos \alpha \cos \beta + \sin \alpha \sin \beta \sin C
\]

On the equatorial aspect, tangent point $T$ lies on the intersection of Equator and an arbitrary central meridian. The projected point $P$ marks a shaded triangle, whose vertices define central angle $\alpha$, latitude $\varphi$ and longitude $\lambda$.

\[
\cos \alpha = \cos \varphi \cos \lambda + \sin \varphi \sin \lambda \cos \frac{\pi}{2} \Rightarrow \alpha = \arccos(\cos \varphi \cos \lambda)
\]

\[
\frac{\sin \theta}{\sin \varphi} = \frac{\sin \pi/2}{\sin \alpha} \Rightarrow \sin \theta = \frac{\sin \varphi}{\sin \alpha}
\]

\[
\cos \varphi = \cos \alpha \cos \lambda + \sin \alpha \sin \lambda \cos \theta \Rightarrow \\
\cos \theta = \frac{\cos \varphi - \cos \alpha \cos \lambda}{\sin \alpha \sin \lambda} = \frac{\cos \varphi (1 - \cos^2 \lambda)}{\sin \alpha \sin \lambda} = \frac{\cos \varphi \sin \lambda}{\sin \alpha}
\]

The Equatorial Azimuthal Equidistant

Just substituting results for $\cos \theta$ and $\sin \theta$,

\[
\rho = r = aR \\
x = aR \cos \varphi \sin \lambda / \sin \alpha \\
y = aR \sin \varphi / \sin \alpha
\]

If $\lambda = \varphi = 0$, $\sin \alpha = 0$, $x = y = 0$.

The Equatorial Azimuthal Equal-Area

In equations for Lambert's equal-area azimuthal projection, substitute $\alpha$ for $\Phi_1$:

\[
x = \rho \cos \theta = \rho \cos \varphi / \sin \lambda \sin \alpha \\
y = \rho \sin \theta = \rho \sin \varphi / \sin \alpha
\]
\[ \sin(a + b) = \sin a \cos b + \cos a \sin b, \] therefore \[ \sin 2a = 2 \sin a \cos a \]

\[ \cos(a + b) = \cos a \cos b - \sin a \sin b; \] \[ \cos 2a = \cos^2 a - \sin^2 a = 2 \cos^2 a - 1, \] therefore \( \frac{\cos(2a + 1)}{2} = \cos^2 a \)

\[ \frac{\rho}{\sin \alpha} = \frac{2R \sin \frac{\alpha}{2}}{\sin \alpha} = \frac{2R \sin \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = \]

\[ \frac{R}{\cos \frac{\alpha}{2}} = \frac{R}{\sqrt{\cos \alpha + 1}} = \frac{R \sqrt{2}}{\sqrt{\cos \varphi \cos \lambda + 1}} \]

\[ x = \frac{R \sqrt{2} \cos \varphi \sin \lambda}{\sqrt{1 + \cos \varphi \cos \lambda}} \quad y = \frac{R \sqrt{2} \sin \varphi}{\sqrt{1 + \cos \varphi \cos \lambda}} \]

Combined azimuthal equidistant (top) and equal-area (bottom) equatorial maps

Again, both projections are very similar near the tangent point: the northern and southern portions of Africa join almost seamlessly.

**A Simple Projection plus Two Derived Works**

**Deducing the Equidistant Cylindrical Projection**

Suppose an arbitrary projection in whose equatorial aspect:

- All meridians are **standard** equally-spaced vertical lines
- All parallels are horizontal, equally-spaced, equally-long lines

The rectangular result is the very simple cylindrical equidistant projection (**equidistant** only along meridians and two parallels), having a **multitude of alternate names**. It is neither **conformal** nor **equal-area**, and despite the resemblance to the **stereographic cylindric**al, it is not truly created by a perspective method.
Since scale is constant along meridians, $y$ is simply $R\varphi$; two parallels are standard at $\pm \varphi_0$, with circumference $R \cos \varphi_0$.

Constant scale along equal-length parallels means:

- $x = R\lambda \cos \varphi_0$
- $y = R\varphi$

Different standard parallels only change the map's width/height ratio. For the common special case of standard Equator (usually known as the *plate carrée*), $\cos \varphi_0 = 1$, and latitude and longitude are directly mapped into $y$ and $x$ respectively, therefore a world map is a 2:1 rectangle.

### Deducing the Winkel I and Eckert V Projections

The cylindrical equidistant projection can be calculated quickly and presents a few interesting properties like immediate determination of angular and linear distances from two points. However, it creates extreme horizontal stretching along the poles. On the other hand, the *sinusoidal projection* is difficult to read at the polar regions due to high shearing. Both have constant scale along equally-spaced parallels.

The **Winkel I** is an arithmetic average of the two preceding projections. It is neither equal-area nor conformal and is defined as:

- $x = R\lambda(\cos \varphi_0 + \cos \varphi) / 2$
- $y = R\varphi$

The **Eckert V** projection is a special case for $\varphi_0 = 0$. Winkel instead chose a standard parallel yielding a map with area in scale with its width. For the upper right quadrant of the map, the right edge is
Deducing the Winkel II Projection

For his second proposal, Winkel used an arbitrary projection in whose equatorial aspect:

- parallels are equally-spaced horizontal lines
- meridians are equally-spaced elliptical arcs
- the whole map fits a 2:1 ellipse

Resembling Mollweide's (but not equal-area), this auxiliary elliptical projection is essentially an extension of Apian's second globular design, defined as:

\[ x = f(y) = \frac{\pi R \cos \varphi_0 + \cos \varphi}{2} \]
\[ y = \varphi R \Rightarrow dy = R \, d\varphi \]

\[ 0 \leq y \leq \pi R/2, 0 \leq \varphi \leq \pi/2 \]

\[ S = \frac{\pi R^2}{2} \int_0^{\pi/2} \left( \cos \varphi_0 + \cos \varphi \right) dy = \]
\[ \frac{\pi R^2}{2} \int_0^{\pi/2} \left( \cos \varphi_0 \int_0^{\pi/2} d\varphi + \int_0^{\pi/2} \cos \varphi d\varphi \right) = \]
\[ \frac{\pi R^2}{2} \left( \varphi \cos \varphi_0 \bigg|_0^{\pi/2} + \sin \varphi \bigg|_0^{\pi/2} \right) = \]
\[ \frac{\pi R^2}{2} \left( \pi/2 \cos \varphi_0 + 1 \right) \]

If area of sphere is \(4\pi R^2 = 4S\),

\[ \pi R^2 = \frac{\pi R^2}{2} \left( \frac{\pi}{2} \cos \varphi_0 + 1 \right) \Rightarrow 1 = \frac{\pi}{2} \cos \varphi_0 \Rightarrow \varphi_0 = \arccos \frac{2}{\pi} \]

or about ±50°27'35'.
The Winkel II projection is a simple arithmetical average of this elliptical projection and the cylindrical equidistant; it is neither conformal nor equal-area. Again, ±50°27"35' is chosen as the standard parallels.

Equation ellipse centered on origin, major axis $a$, minor axis $a/2$:
\[
\frac{x^2}{a^2} + \frac{4y^2}{a^2} = 1
\]

On the first quadrant, the map edge is $x = \sqrt{a^2 - 4y^2}$
Equally-spaced parallels mean $y = R\varphi$, therefore $a = \pi R$
Since meridians are equally-spaced,
\[
x = \frac{\lambda}{\pi} \sqrt{\pi^2 R^2 - 4 R^2 \varphi^2} = \frac{\lambda R}{\pi} \sqrt{\pi^2 - 4 \varphi^2} \\
y = R\varphi
\]

The Winkel II projection is a simple arithmetical average of this elliptical projection and the cylindrical equidistant; it is neither conformal nor equal-area.
\[
x = \frac{R\lambda}{2} \left( \cos \varphi_0 + \frac{\sqrt{\pi^2 - 4 \varphi^2}}{\pi} \right) \\
y = R\varphi
\]

Again, ±50°27"35' is chosen as the standard parallels.

The underlying elliptical projection is also (erroneously) mentioned to be an unmodified Mollweide ellipse, with non-uniformly spaced parallels.

### Three Modifications for Azimuthal Projections

**Aitoff's Projection**

The equatorial aspect of the azimuthal equidistant projection presents the whole world in the familiar "horizontal" aspect; however, there is significant areal exaggeration near the map boundaries.
Noticing that an azimuthal equidistant map encloses an "inner" hemisphere in a disc whose radius is half that of the whole map, Aitoff proposed a very simple, yet attractive modification:

1. project the world with doubled longitudinal coordinates, effectively cramming everything into the inner hemisphere
2. double the horizontal scale, stretching the disc into a 2:1 ellipse

The resulting projection, no more azimuthal, is equidistant only along the Equator and central meridian.

Projection equations follow directly from those for the equatorial azimuthal equidistant, substituting $\lambda / 2$ for $\lambda$ and multiplying a factor 2 in $x$ coordinates:

\[
\alpha = \arccos \left( \cos \phi \cos \left( \frac{\lambda}{2} \right) \right) \\
x = 2aR \cos \phi \sin \left( \frac{\lambda}{2} \right) / \sin \alpha \\
y = aR \sin \phi / \sin \alpha
\]

Hammer's and Eckert-Greifendorff's Projections

Aitoff's work was itself modified by Hammer, whose projection applied the same idea, but to Lambert's azimuthal equal-area projection instead. As a consequence:

- global scale is smaller than in Aitoff's
- the projected inner hemisphere is not half as wide as the whole map, but encloses half its area
- the final doubling restores scaled proportions and the final map is also equal-area
- scale is no more constant along the major axes

Again, formulas can be deduced replacing $\lambda$ by $\lambda / 2$, this time in Lambert's:

\[
x = \frac{2R \sqrt{\cos \phi} \sin \frac{\lambda}{2}}{\sqrt{1 + \cos \phi \cos \frac{\lambda}{2}}} \\
y = \frac{R \sqrt{\sin \phi}}{\sqrt{1 + \cos \phi \cos \frac{\lambda}{2}}}
\]
Scales are different but overall lines are fairly similar in Aitoff and Hammer projections. Since differences in meridian spacing are hardly visible in the inner hemisphere, these two projections were frequently mislabeled.

Hammer’s design was in turn modified by Eckert-Greifendorff, in a projection applying a further 2 : 1 rescaling. Therefore equations are identical, except for substituting $\lambda / 4$ for $\lambda / 2$ and changing the $x$ factor from 2 to 4.

**Winkel’s Tripel Projection**

Yet another modification of Aitoff’s projection was devised by Winkel. Much like in his first and second hybrid maps, his tripel projection averages the equidistant cylindrical projection, this time with Aitoff’s. Again, $\varphi_0 = \pm \arccos 2/\pi$ are usually chosen as the standard parallels for the cylindrical base (although the final projection has no standard parallels). Equations follow directly from Aitoff’s and the equidistant cylindrical’s:

\[
\begin{align*}
\alpha &= \arccos (\cos \varphi \cos (\lambda/2)) \\
w &= 0 \text{ if } \sin \alpha = 0, \ 1 / \sin \alpha \text{ otherwise} \\
x &= R(\lambda \cos \varphi_0 + 2w \alpha \cos \varphi \sin (\lambda/2)) / 2 \\
y &= R(\varphi + w \alpha \sin \varphi) / 2
\end{align*}
\]
Azimuthal Projections

Introduction

Given a reference point A and two other points B and C on a surface, the azimuth from B to C is the angle formed by the minimum-distance lines AB and AC (which, on a sphere, are geodesic or great circle arcs). In other words, it represents the angle one sitting on A and looking at B must turn in order to look at C. The bearing from A to C is the azimuth considering a pole as reference B.

All azimuthal projections preserve the azimuth from a reference point (the conceptual center of the map), thus presenting true direction (but not necessarily distance) to any other points. They are also called planar since several of them are obtained straightforwardly by direct perspective projection to a plane surface.

Orthographic  Stereographic

Vertical perspective  Gnomonic

Schematic cross-section of development of some perspective azimuthal projections in a polar aspect. Red light rays shoot from latitudes 0°, 30°, 45°, 60° and 90° to the blue projection plane. By construction, not every point can be represented.

Compared polar aspects of five azimuthal projections with parallels spaced 10° apart. Orthographic and stereographic stop at Equator, gnomonic is arbitrarily clipped at 20°. Equatorial zone is red, polar caps blue.

The polar aspect is easily built for azimuthal projections; one of the poles is the central point, making the graticule trivial:

- meridians are straight lines, radiating regularly spaced from the central point
- parallels are complete concentric circles
- projections are only distinguished by parallel spacing

In any aspect, all straight lines touching the central point are geodesics, and distortion depends only on distance from the center.
In a few two-point azimuthal projections, correct angles are presented from two specific locations instead of one.

While azimuthal maps quickly tell the direction to anywhere from the central point, retroazimuthal projections have the opposite property, showing the correct direction to turn from any place to the central point.

**Classic Azimuthal Projections**

Among the oldest projections, three classic azimuthal designs are defined by pure geometric perspective constructions.

**Azimuthal Orthographic Projection**

Used by the Greek Hipparchus (2nd century B.C.), but probably known earlier, this projection was called analemma by Ptolemy and renamed orthographic by d'Aiguillon (1613).

The orthographic projection is mainly used for (sometimes dramatic) illustration purposes since it clearly shows the Earth as seen from space infinitely far away, thus closely matching a student's view of a globe.
Severe shape and area distortion near the map borders prevents its general use for world maps.

Its construction can be easily explained and compared to other azimuthal projections's.

**Azimuthal Stereographic Projection**

![Stereographic map, central meridian 110°W](image1)

![Stereographic map, central meridian 70°E](image2)

Probably the most widely used azimuthal and known since Classical eras, this projection usually attributed to Hipparchus was called *Planisphaerum* by Ptolemy and *stereographic* by d'Aiguillon (1613). It is a conformal projection: over a small area, angles in the map are the same as the corresponding angles on Earth's surface. It also preserves circles, no matter how large (great circles passing on the central point are mapped into straight lines), although concentric circles on the sphere will not generally remain concentric on the map. On the other hand, the loxodrome is plotted as a logarithmic spiral.

An azimuthal stereographic map has a simple geometric interpretation: rays emanating from one point pierce the Earth's surface hitting a plane tangent at the antipode. The result is the map backface, which covers the entire plane (regions near the source point lie at infinity, and that point itself cannot be mapped).

Because - in contrast to the azimuthal orthographic - scale is greatly stretched away from the center of the map, azimuthal stereographic maps are usually constrained to the hemisphere opposite the source point, or an even smaller region.

The azimuthal stereographic was also modified for the ellipsoidal case; conformality is maintained, but the result is no more exactly azimuthal or circle-preserving. In this form, it is part of the UTM grid.

**Gnomonic Projection**
The gnomonic (also called central) projection is constructed much like the azimuthal stereographic, but the ray source is located exactly on the sphere's center; therefore it can present even less than one hemisphere at a time. Distance distortion is pronounced except very near the tangent point.

This unique projection's most important property is that every geodesic, including the Equator and all meridians, is mapped to a straight line, making easy finding the shortest route between any two points (but not the direction to follow).

**Nonperspective Azimuthal Projections**

Unlike the "classical" (orthographic, stereographic and gnomonic) designs, azimuthal projections like the equidistant and equal-area were derived **mathematically** without a real perspective process. Both can map a full sphere, with an "inner" hemisphere surrounded by a ringlike "outer" one. However, for lesser overall distortion the latter may be presented in a separate map centered on the antipodal point.

**Azimuthal Equidistant Projection**
Able to present the whole Earth in a single map and with constant radial scale (distances increase linearly from the center of projection), the azimuthal equidistant projection is further discussed elsewhere due to its important features.

In the north polar aspect, the azimuthal equidistant is familiar as part of both flag and emblem of the United Nations Organization, with olive branches replacing Antarctica. The austral continent, here turned "inside-out", illustrates this projection's extreme distortion of shapes and areas far from the center.

Simple in construction, this projection is sometimes clipped to a single hemisphere, or even restricted to insets for polar caps.

**Lambert's Azimuthal Equal-area Projection**

Like the superficially similar azimuthal equidistant, the azimuthal projection published by Johann H. Lambert in 1772 strongly distorts shapes in the boundary of a worldwide map. However, radial scale is not
constant: in the polar aspects, parallels get closer together towards the border. As a result, the map preserves areas.

Relatively simple in construction, this projection is frequently used in all aspects.

**Cylindrical Projections**

**Introduction**

In the equatorial (the most common, and frequently the only useful) aspect of all cylindrical projections:

- all coordinate lines are straight
- parallels (by convention horizontal) cross meridians always at right angles
- scale is constant along each parallel, so meridians are equally spaced
- all parallels have the same length; the same happens to meridians

Therefore,

- whole-world maps are always rectangular
- scale is identical in any pair of parallels equidistant from Equator
- scale differs considerably among parallels, reaching infinity at poles, which have zero length on the Earth but are as long as the Equator on a cylindrical map

On the transverse aspect, two opposite meridians lie over the Equator of the equatorial aspect; other properties don't hold.

Rolling a rectangular map and joining two opposite edges creates a tube, or a cylinder without end caps. In fact, some cylindrical projections are geometrically derived from closely fitting a tube around a sphere; the former may be secant or tangent, and as a result two parallels or the Equator, respectively, are standard lines.

All cylindrical projections are remarkably similar, being in fact only distinguished by parallel spacing. The very important, unique conformal cylindrical projection is named after Mercator and discussed elsewhere. There is a single equal-area cylindrical projection, not counting rescaled versions.

As a group, cylindrical projections are more appropriate for mapping narrow strips centered on a standard parallel. Although useful for comparison of regions at similar latitudes, they are badly suited for world maps because of extreme polar distortion. Unfortunately, cylindrical maps
are often employed in textbooks and other popular publications, perhaps due to poor research and their simple shape neatly fitting page frames.

**Equidistant Cylindrical and Cassini Projections**

Equidistant cylindrical map with 0° as standard parallel, also called *plate carrée*.

Cassini map, a transverse aspect of the *plate carrée*. Central meridians 70°E and 110°W. Like the normal aspect, two shorter edges may be joined forming a tube.

The simplest of all map graticules belongs to the equatorial aspect of the equirrectangular projection, referred to by many names like equidistant cylindrical, plane chart, plain chart and rectangular. It is a cylindrical projection with standard meridians: all have constant scale, equal to the standard parallels's, therefore all parallels are equally spaced. It was credited to Erathostenes (ca. 200 b.c.) and to Marinus of Tyre (ca. 100). Its trivial construction made it widely used, even for navigation, until the Modern Age.

A special case of the equirrectangular projection is called *Plate Carrée*, or simple cylindrical: the Equator is a standard parallel, so it is twice as long as all meridians, making the map a 2 : 1 rectangle and the graticule's grid square.

Fast, trivial equations led to its resurgence in rough computer-drawn
maps, with early machines or real-time graphics. It is still commonly used in digitized textures ("skins") of earthly and planetary features.

A transverse aspect of the equidistant cylindrical projection was proposed by César F. Cassini in 1745, and named after him. Cassini's grandfather, Jean Dominique (born Giovanni Domenico) was the most prominent member of a family of astronomers and cartographers. Several European countries used Cassini's projection in large-scale topographical maps until recently. Distortion is identical as in the normal aspect, thus a central meridian, the Equator, and three other meridians at multiples of 90° are all straight lines with equal, constant scale.

Seldom found, oblique equidistant cylindrical maps are useful for quickly calculating angular and linear distances from two points on the map to any other point (Botley, 1951).

**Miller's Cylindrical Projection**

The best-known (1942) of all projections published by Osborn M. Miller was mathematically conceived as a compromise to Mercator's, retaining its familiar shapes but with much smaller polar exaggeration. It applies a reduction factor of 0.8 to the latitudes before calculating Mercator's equations, and an inverse factor to the result. In consequence, the map can include the whole world. The projection is neither conformal nor area-preserving.

Miller created several other projections, including three other cylindrical designs; none was as popular as the one bearing his name.
Some cylindrical projections are defined by a geometric process based on perspective. It can roughly be imagined as a semitransparent spherical shell wrapped by a tube, secant or tangent. While both sphere and tube slowly rotate around the latter's axis, a fixed source shoots light rays along a single meridian, projecting "shadows" of spherical features onto the tube. After a complete revolution, the tube is cut along a line parallel to its axis and unrolled.

Just by changing the source position and tube's diameter, different maps result. The source may be located infinitely away, making rays parallel.

In contrast, other cylindrical projections like the equidistant cylindrical, Miller and Mercator have conventional graticules defined arbitrarily, not by a light source analogy.

**Lambert's Cylindrical Equal-Area Projection**
Sometimes mentioned in connection to Archimede, the equal-area projection on a tangent cylinder was rigorously defined by Johann H. Lambert in both equatorial and transverse aspects, among several other projections (1772). It preserves areas, but only the Equator is free of shape distortion.

This projection's perspective is easily visualized by rolling a flexible sheet around the globe and projecting each point horizontally onto the tube so formed. In other words, light rays shoot from the cylinder's axis towards its surface, which is afterwards cut along a meridian and unrolled.

Like most cylindrical projections, it is quite acceptable for the tropics, but practically useless at polar regions, which are rather compressed, resulting in a map much broader than tall. Again like in other cylindrical projections, deformation is uniform along the same parallel.

Gall's Orthographic Projection
James Gall's orthographic equal-area projection (1855) is trivially similar to Lambert's version, but with standard parallels at 45°N and 45°S. Therefore, the projection cylinder is secant and narrower; the vertical amplitude must be proportionally expanded in order to preserve the total mapped area. Thus, the only real difference is the aspect ratio (i.e., width divided by height): Gall's orthographic is twice as tall.

Although area-preserving, this projection's unconventional pattern of shape distortion limits its usefulness.

"Peters" (Gall-Peters) Projection

In 1967, Arno Peters published a projection essentially identical to Gall's orthographic version. Perhaps its creation was actually independent; however, Peters kept claiming the projection as a personal creation and a novelty even after heavy criticism by knowledgeable cartographers. After 1973, the projection was heavily promoted and gained widespread press coverage on the following, very exaggerated, grounds:
- free of extreme shape distortion (however, not only there's severe distortion but the distortion pattern greatly changes along each meridian)
- free of distance distortion (true only along the two standard parallels)
- equal-area (true, but several other equal-area maps had been created previously, many without the severe distortions above)
- more "egalitarian" than the Mercator and other cylindrical projections at the time, presenting Third-World nations in their proper size

Equal-area cylindrical projections compared at identical scales. Standard parallels are outlined.

Lambert (1772); standard latitude 0°, aspect ratio 3.141:1

W. Behrmann (1910); standard latitude 30°, aspect ratio 2.356:1

Trystan Edwards (theoretical constraint, 1953); standard latitude 37°24', aspect ratio 1.983:1

Hobo-Dyer (2002); standard latitude 37°30', aspect ratio 1.977:1

Gall (1855), Peters (1967); standard latitude 45°, aspect ratio 1.571:1
This last argument actually helped Peters (who studied political propaganda) winning the preference of several organizations like UNESCO for their printed maps. This is unfortunate, for projections must not be chosen due to a single feature or on the basis of publicity alone, no matter how sympathetic the cause. Besides, there are many equal-area projections much more suited to general-purpose world maps than Gall-Peters's.

Other Equal-area Cylindrical Projections

Lambert's principle is employed by a few lesser-known equal-area cylindrical projections, changing only standard parallels and therefore general map proportions. Each of them can be converted to any other simply by rescaling both width and height.

Their patterns of shape distortion are similar and, like all cylindrical projections, independent of longitude: vertical scale is more affected the farther from the standard parallels (exaggerated between them and compressed in the outer portions).

Some of these variants were explicitly designed in order to reduce maximum or average deformation (as conveniently defined by the author), as is the case of Behrmann's and Trystan Edwards's projections. Notably, for some reason the latter specified a deformation criterion whose standard parallel does not match the value actually chosen.

Stereographic Cylindrical Projections: Gall's and Braun's
The geometric construction for James Gall's preferred projection (1885) resembles the perspective for the **azimuthal stereographic**, with two differences:

- the projection surface is a secant cylinder 45°N and 45°S, like in his orthographic
- the ray source for every projected point sits at the Equator on the opposite meridian

Areas are not preserved and the map is not **conformal**. Scale is true only along the **standard** parallels 45°N and 45°S. There are no outstanding features except overall distortion.
Carl Braun's stereographic cylindrical (1867) is a very similar projection, but developed from a tangent cylinder.

**Central Cylindrical Projection**

In the central cylindrical (also called centrographic cylindrical) projection, vertical scale increases very fast far from the map's centerline, even faster than in Mercator's projection; likewise, poles cannot be shown in the equatorial aspect.

Its origin is unknown, though it has an obvious analogue in the azimuthal gnomonic projection. With no favorable property, neither equal-area nor conformal, it is almost never used, either in equatorial or transverse (called the Wetch projection, after J.Wetch, 19th century) aspects.

**Pseudocylindrical Projections**

**Introduction**

In all cylindrical projections there is strong shape distortion, and usually area is also greatly exaggerated, at higher latitudes (in the normal aspect). In particular, the poles are infinitely stretched to lines, or can not even be included, as in Mercator's projection. The pseudocylindrical
class of projections attempts another trade-off of shape vs. area; in the normal equatorial aspect, they are defined by:

"Polycylindrical" concept presented by nine strips with partial cylindrical equidistant maps (parts of globe and maps removed for clarity). Each fits a different standard parallel, therefore horizontal scale varies, but vertical scale is identical in all maps.

- straight horizontal parallels, not necessarily equidistant
- arbitrary curves for meridians, equidistant along every parallel

Some properties, inherited from cylindrical maps, are noteworthy:

- horizontal parallels visually preserve latitude relations, thus easing correlation of phenomena which mostly depend on distance from Equator, e.g., daylight periods, climate, winds and greenhouse warming
- constant scale at any point of a parallel eases measurements in its direction

Since their parallels and meridians do not always cross at right angles, conformality is denied to pseudocylindrical maps; in fact, most suffer from strong shape distortion at polar regions. Therefore, many were designed for equivalence.

Pseudocylindrical projections could be called "polycylindricals", by analogy to polyconic projections being a generalization of the conic group. Indeed, any pseudocylindrical map can be conceptually created by juxtaposing a (possibly infinite) number of partial cylindrical maps.
Of infinitely many possible pseudocylindrical projections, several are useful didactical devices and popular choices for world maps.

**Common Pseudocylindrical Projections**

**Sinusoidal (Sanson-Flamsteed) Projection**

![Sinusoidal (Sanson-Flamsteed) map, graticule spacing 10°](image)

Despite the common name, this projection was not first studied by either Sanson (ca. 1650) or Flamsteed (1729, posthumously), but probably by Mercator (at least it was included in later editions of Mercator’s atlas). It is also called *sinusoidal* and Mercator equal-area, and can be easily deduced. It preserves area and also distances along the horizontals, i.e., all parallels in a normal map are standard lines. Although the equatorial band is reproduced with little distortion, polar caps can be hard to read. The partially constant scale and simple construction still recommend this projection for continents like Africa and South America, frequently after convenient recentering.

Both the Equator and central meridian are standard lines, thus the whole map is twice wide as tall.

**Mollweide Projection**

![Mollweide map](image)

Created in 1805 by Karl Mollweide from Germany, and popularized by Jacques Babinet in 1857, this equal-area projection was designed to inscribe the world into a 2:1 ellipse, keeping parallels as straight (but not
standard) lines while still preserving areas. All meridians but the central one map to elliptical arcs. This projection is also called *homalographic,* *homolographic* (Greek *homo* "same"), *elliptical* or *Babinet,* and its *interrupted* form was popularized by Goode.

In a sense, the sinusoidal and Mollweide projections handle polar regions in complementary ways: while the first overcrowds them, the latter creates widely spaced meridians, with much greater angular distortion. These two designs were often combined in *hybrid* approaches like Goode's *homolosine projection.*

**Collignon Projection**

Édouard Collignon's projection, introduced in 1865, preserves areas but strongly distorts shape. In the equatorial aspect, both northern and southern hemispheres can be either a isosceles triangle with base on the Equator and height half the Equator length, or a isosceles trapezium (British: trapezoid) with the small base at the Equator. All graticule lines are straight but the meridians are optionally broken at the Equator.

The two most common arrangements for the worldwide map are either the isosceles triangle (with unbroken meridians and a flat South Pole for base) or the diamond. The two complementary options (upside-down triangle and hourglass-shaped) and the *interrupted* variant in two or more diamonds are equally valid. In spite of its simple construction, this projection is regarded as little more than a curiosity.

**Quartic Authalic Projection**
The quartic authalic is another equal-area (authalic from Greek *autos ailos*, "same area") projection, designed by Karl Siemon (1937) and independently by Adams (1944). Its meridians are fourth-order curves, hence the name.

**Flat-Polar Pseudocylindrical Projections**

Maps with polelines, called flat-polar, represent the poles as straight lines instead of points. Of course all cylindrical projections are flat-polar in the equatorial aspect, but the term is commonly applied to a large group of pseudocylindrical designs. Cartographers such as K. H. Wagner and M. Eckert developed whole "families" of flat-polar projections.

Polelines avoid the crowded appearance of projections like the sinusoidal and Mollweide at the cost of scale distortion.

**Flat Polar Quartic Projection**

Several projections created by Felix W. McBryde and Paul Thomas have polelines one-third as long as the Equator. The fourth and best known (1949) has fourth-order curves as meridians and is equal-area.

**Flat-polar Projections by Eckert**
In odd-numbered Eckert projections (Eckert I here), the graticule is nearly square at the center.

In 1906 the German professor Max Eckert (later Eckert-Greifendorff) published six pseudocylindrical projections sharing some features in the normal aspect:

- Eckert’s projection II is similar to his first design, but nonconstant parallel spacing makes it equal-area.
  - the central meridian is straight, half as long as the Equator, and a standard line in odd-numbered projections.
  - poles are flat, half as long as the Equator.
  - even-numbered projections are equal-area.
  - odd-numbered projections have equally spaced parallels.

Therefore, in all six proposals the poles are framed by a square, and the whole map by a rectangle twice as broad. The boundary meridians are simple curves.
Although none of the six is conformal, the odd-numbered projections present a better overall shape (there’s no shape distortion at the very center); in order to preserve area, the even-numbered projections compress vertical scale near the poles and stretch it near the Equator.

Eckert's second design is equal-area and maps all meridians to straight lines broken at the Equator. The first projection is similar, but not equal-area since the parallels are equally spaced. Neither is more than a curiosity.

For his third and fourth projections, Eckert made the outer meridians as half circles; all other meridians are regularly spaced elliptical arcs except the central which, like in all Eckert flat-polar maps, is straight and half as long as the Equator. The fourth design was moderately used for world maps; the third is sometimes mistaken for Ortelius's oval map, which has not constant scale along parallels.
The sixth and most popular of Eckert’s flat-polar projections has boundary meridians shaped as half the period of a sinusoid. The superficially similar fifth design has regularly spaced parallels and is not equal-area.

Several other pseudocylindrical projections, most notably by Siemon, McBryde and Thomas, and a series by Wagner, are also based on polelines and sinusoidal meridians.

**Robinson Projection**

Following a widespread controversy about the adequacy of cylindrical world maps for teaching, Rand McNally, the traditional atlas publisher, requested the distinguished cartographer and educator Arthur H. Robinson to develop a new map projection with reduced overall distortion and a simple, uninterrupted graticule.
Instead of using a perspective process such as that used for classic azimuthal projections, or simple mathematical functions like the sinusoidal or elliptical arcs of the Sanson-Flamsteed, Mollweide and Eckert's series, the resulting compromise projection had the boundary meridians in the equatorial aspect defined by conventional values, calculated by hand in order to yield a "right-looking" map (thus its common name, orthophanic). A table defines $x$, $y$ coordinate values for 5° increments of latitude for those meridians; other points must be interpolated. Like in all pseudocylindrical projections, meridians are equally spaced along all horizontal, straight parallels (regularly spaced between 38°N and 38°S). The Equator is nearly twice as long as the central meridian; poles are flat.

Designed in 1963 and formally published in 1974, the Robinson projection became truly popular only after praised by the cartographic staff of the National Geographic Society; in 1988, it was published as an insert of the Society's magazine and chosen as its reference world map, replacing the van der Grinten projection.

Some Hybrid Pseudocylindrical Projections

Several authors have attempted to combine advantages of two or more existing projections. Values can be mathematically averaged, or different pieces of the map may be separately projected along lines of similar scale. This latter approach is made convenient in pseudocylindrical projections due to their straight parallels with constant scale; therefore different map "slices" can be projected separately, then stitched together, possibly after rescaling.

Goode Homolosine
John P. Goode combined the sinusoidal and Mollweide (homolographic) projections in his hybrid homolosine (homolographic + sinusoidal) projection of 1923-25: three horizontal stripes are joined at the two parallels with the same length in the two previous projections. Latitudes higher than approximately 40°44‘12”N and 40°44‘12”S are represented using Mollweide's projection, and the remaining area by the sinusoidal. The meridians are broken at the joint and the result is not appreciably better than either original methods used alone; however, horizontal scale is preserved in nearly 65% of the map and the polar caps are reasonably legible while preserving the sinusoidal's constant meridian scale at the center. This projection, especially designed for interruption, was for long quite popular in atlases.

**Boggs Eumorphic**

Created by S.W. Boggs, the eumorphic (Greek for "well-shaped") projection of 1929 is another hybrid. However, instead of discretely joining separate bands, it defines its y-coordinates as the arithmetic average of corresponding sinusoidal and Mollweide coordinates. The x-coordinates are calculated for an equal-area map, usually presented in interrupted form.
Eckert’s fifth proposal (1906) can be built as an arithmetic mean of the sinusoidal and plate carrée projections (actually the y-coordinates are the same in both). As a result, poles are mapped to straight lines with half the Equator’s length. All the meridians but the central are sinusoids. Although the map does not preserve areas, it deceptively resembles the much more popular equal-area Eckert VI projection.

Winkel I

The first projection published by O. Winkel in 1921 is a generalization of Eckert’s V using the equidistant cylindrical projection with any two opposite parallels standard, not necessarily the Equator (therefore only the horizontal scale is changed from the special case). Winkel preferred 50°27"35' N and S, which makes the total area proportional to the equatorial circumference.

Winkel II

Also published in 1921, this projection averages the equidistant cylindrical and a 2:1 elliptical projection similar to Mollweide’s, but with equally-spaced parallels and therefore not equal-area.
The resulting map, also neither conformal nor equal-area, is constructed much like Winkel's first projection.

Conic Projections

Introduction

Map wrapped on a cone

Conic projections, in the normal polar aspect, have as distinctive features:

- meridians are straight equidistant lines, converging at a point which may or not be a pole. Compared with the sphere, angular distance between meridians is always reduced by a fixed factor, the cone constant
- parallels are arcs of circle, concentric in the point of convergence of meridians. As a consequence, parallels cross all meridians at right angles. Distortion is constant along each parallel

For illustration purposes, the resulting shape can be wrapped on a cone set atop the mapped sphere, although very few conic projections are based on true geometric perspective (in other words, the cone is always the result, but seldom directly participates in its construction). Typically the cone intersects the sphere at one or two parallels, chosen as standard lines.

Due to simple construction and inherent distortion pattern, conic projections have been widely employed in regional or national maps of temperate zones (while azimuthal and cylindrical maps were favored for polar and tropical zones, respectively), especially for areas bounded by two not too-distant meridians, like Russia or the conterminous United States. On the other hand, conic projections are seldom appropriate for world maps.

Relatively few projections are called "conic"; nevertheless, many others are ruled by conic principles, since the cone is a limiting case of both the circle (a cone with no height, and cone constant 1) and the cylinder (a cone with vertex at infinity, with standard parallels symmetrical north and
south of the Equator). There is only one type of **equal-area** conic projection, and only one is **conformal**.

Conic constraints are relaxed by **pseudoconic** (with curved meridians) and **polyconic** (with nonconcentric parallels) projections. Conic and coniclike are among the oldest projections, being the base for Ptolemy's maps (ca. 100).

**Equidistant Conic Projections**

Equidistant conic map, standard parallels 60°N and Equator, central meridian 0°. A full map is presented for illustration only, since this projection is seldom used for worldwide maps.

Equidistant conic map, standard parallels 30°N and 60°S

Euler map, limiting parallels 90°N and Equator, standard parallel 45°
The equidistant (also called \textit{simple}) conic projection has constant parallel spacement, thus scale is the same along all meridians. Commonly one or two parallels are chosen to have the same scale, suffering from no distortion.

Neither equal-area nor conformal (but an acceptable compromise for most temperate countries), this projection is defined arbitrarily instead of by a perspective process. It is the general case of both \textit{azimuthal equidistant} and \textit{equidistant cylindrical} projections.

Historically, Ptolemy's first known map resembled an equidistant conic with meridians broken at the Equator. The equidistant concept was adopted and "improved" by many authors, the most famous being Joseph N. de l'Isle (also Delisle), one of a family of cartographers and map publishers. De l'Isle introduced variations to the equidistant model; the projection today bearing his name (ca. 1740) is not a true conic, despite a close resemblance. B. Mead proposed in 1717 a variation with parallels comprising 1°-long straight segments.

Several cartographers kept the general arrangement but studied criteria for standard parallel placement in order to minimize distortion. These include a series by P. Murdock (1758) and Everett (1903), and Euler's projection (1777). Russian usage of the equidistant conic led to other derivations, including by V. V. Kavrayskiy (ca. 1930, 62°N and 47°N) and by D. Mendeleyev (1907, 90°N and 55°N) of chemistry fame.

\textbf{Equal-area Conic Projections by Lambert and Albers}
For the geometric construction of the equal-area conic projection published by the German Heinrich C. Albers (1805), light rays emanate from the globe surface and hit the cone as a normal from its surface. As usual, there is little distortion along the central parallel and none on the standard ones. The standard parallels may lie on different hemispheres, but if equidistant from the Equator, the projection degenerates into an equal-area cylindrical.

This projection was commonly applied to official American maps after usage of the polyconic projection declined.

In a particular case of Albers's conic projection, either 90°N or 90°S is chosen as a standard parallel, and therefore meridians converge at a pole. Published by Lambert in 1772, this projection preserves areas, thus parallels are farther apart near the vertex, getting closer together towards the non-standard pole. When 0° is chosen as the other standard parallel, the result is a cone constant of 1/2 and a semicircular map. Lambert himself chose a constant of 7/8 for his map of Europe: the resulting standard parallel, roughly 48°35’N, lies between Paris and Munich.

This projection was employed much less frequently than Albers's. In fact, it is probably the least known of Lambert's projections.
Lambert's Conformal Conic Projection

Conformal conic map with standard parallels 50°N and 10°S, clipped at 50°S.

The same paper (1772) with Lambert's equal-area conic projection included his conformal conic design: Lambert explicitly investigated a conic approach as intermediary between the then known conformal projections, azimuthal stereographic and Mercator's. These are in fact special cases of the conformal conic, obtained respectively when one pole is the single standard parallel and when the standard parallels are symmetrically spaced above and below the Equator.

This projection remained essentially ignored until World War I, when it was employed by the French military. Since then, it has become one of the most widely used projections for large-scale mapping, second only to Mercator's.

Like in all conformal projections, scale distortion is greatly exaggerated in the borders of a worldwide map, although less than in Mercator's. Meridians converge at the pole nearest the standard parallels; the opposite pole lies at infinity and can not be shown. Scale distortion is constant along each parallel. Meridian scale is less than true between the standard parallels, and greater "outside" them.

Perspective Conic Projections

Braun Stereographic Conic Projection
Actually the only conic projection presented here which is defined by a simple geometric construction, the stereographic projection created by C. Braun (1867) encloses the globe in a cone aligned with the north-south axis, 1.5 times as tall as the globe and tangent at the 30°N parallel. The projection center is the South pole and the resulting map fits a perfect semicircle.

**Polyconic Projections**

Three partial equidistant conic maps, each based on a different standard parallel, therefore wrapped on a different tangent cone (shown on the right with a quarter removed plus tangency parallels). When the number of cones increases to infinity, each strip infinitesimally narrow, the result is a continuous polyconic projection.

Cartographers apply the name **polyconic** to:

- a specific map projection associated with F. R. Hassler, also called American polyconic
- a few projections derived from Hassler's, all geometrically inspired by stacked, overlapping cones; they include the rectangular polyconic
- a more general group of projections with nonconcentric circular parallels in the normal aspect

Quite heterogeneous, the latter group includes many designs with little or no relationship with cones other than the name. They include works by
McCaw, Ginzburg and Salmanova. Some authors extend the definition to include projections like Aitoff’s and Hammer’s.

(American) Polyconic Projection

![Polyconic map, central meridian 100°W, emphasizing its classic use: mapping the United States.](image)

In ordinary conic projections, only one or two parallels, where the conic and spherical surfaces coincide, have correct scale. However, the map may be divided in strips of similar latitude, each fitted to a different cone. Cone constant varies from one at poles to infinity at the Equator, so the strips are not continuous, except along the central meridian. When infinitely many cones are used, each optimally tangent to a thin strip containing a single parallel, the gaps disappear; if the central meridian has constant correct scale, the result is the classic or common polyconic projection, also called American polyconic.

Most authors credit the Swiss Ferdinand R. Hassler with designing the classic polyconic (ca. 1820) while leading the government agency called, for most of its history, the U.S. Coast and Geodetic Survey. Applied in ellipsoidal form to most official large-scale maps until about 1920, it was adopted by several other countries and official agencies.

The classic polyconic projection has circular parallels (except the Equator), all with constant and correct scale, but not concentrical. The same scale applies to the straight central meridian; all other meridians are curved. Neither equivalent nor conformal, this projection is better suited for local or regional maps.

Rectangular (War Office) Polyconic Projection
Also developed (1853) at the U.S. Coast and Geodetic Survey, the best-known modification of Hassler's projection was widely employed for large-scale mapping by the British War Office, thus its common name. It is also called rectangular polyconic due to graticule angles, not overall map shape.

In the rectangular polyconic projection, parallels are circular arcs, again equally spaced along the straight central meridian. However, their scale is not constant, but changes in order to make each meridian cross every parallel at right angles. This is not a sufficient condition for conformality, neither is the result equivalent. Besides, only the Equator (common case) or two parallels symmetrical about the Equator have true length.

Usage of the rectangular polyconic projection is similar to the classic polyconic's; in fact, for small regions they are barely distinguishable.

**Pseudoconic Projections**

In the normal aspect for the artificial group of projections known as pseudoconic, all parallels are circular arcs with a common central point; however, meridians are not constrained to be straight lines, in contrast to true conic projections. The concept is quite old and was used by Ptolemy.
Cordiform Maps

*Stabius-Werner Projections*

Map shapes are not constrained to rectangles, discs or ellipses. Some are not even convex, like those created by the beautiful projections devised (ca. 1500) by Johann Stabius of Vienna, also known as Stab. The three *cordiform* ("heart-shaped") projections were so popularized in treatises by Johannes Werner that they usually bear his name. They all share some features in the normal aspect:

- parallels are arcs of circle centered on a pole (commonly North), all with identical linear scale
- the central meridian is a straight *standard line*
- all other meridians are curves
Only parallel scale distinguishes the three projections. On a worldwide map drawn using the first one, the Equator is a circle and boundary meridians would significantly overlap, therefore the map is normally clipped to one hemisphere. On the third, equatorial scale is slightly larger than the central meridian's, so there is a small overlap north of the 60° parallel.

**The Werner Projection**

The three Stabius-Werner projections are equal-area and clearly suggest the Earth's roundness, much like as its crust were cut at a meridian and peeled off. However, only the second version - known as the Werner projection - was widely used. It has the Equator twice as long as the central meridian, therefore all parallels are standard lines and there is no overlap.

Works by Oronce Finé (1531) and Mercator (1538) employed a butterfly-shaped Werner map interrupted at the Equator, with a central meridian emphasizing the Eastern hemisphere.

This projection is seldom used today in its original shape, but it does appear in some specialized forms, notably interrupted in irregular petals around Antarctica as an inverted star-like map emphasizing oceans, and combined with part of an azimuthal hemisphere for a more conventional star in the "tetrahedral" projection.
A butterfly-shaped, double Werner map, interrupted at the Equator.

Transverse Werner map, interrupted at 25°E

The "Bonne" Projection

Bonne map, central parallel 45°N

Once very popular for large-scale topographic maps, the "Bonne" pseudoconic projection has generally fallen in disuse, been usually replaced by transverse Mercator maps. Although named after the French R. Bonne (1727-1795), it was used much earlier, ca. 1500. It preserves areas, and its shape distortion is acceptable except far from the center.
In a Bonne map all parallels are concentric arcs of circle, all equally spaced and all standard lines. Scale is also correct along the straight vertical central meridian. For construction, one parallel at the sphere is chosen, and a cone tangent at that central parallel is built. The parallel's radius at the map is the same as the radius along the cone. All other parallels's radii are marked accordingly.

As a consequence, each central parallel creates a different Bonne map. Two special cases are well-known:

- the Werner projection, with central parallel 90°N, zero radius and a zero-sized flat "cone"
- the sinusoidal or Sanson-Flamsteed projection, with central parallel 0°. In this particular case, the radius is infinite, the "cone" cylindrical and every parallel just a straight line.

**Modified Azimuthal Projections**

In true azimuthal projections, all directions are preserved from the reference point, usually tangent at the center of the map. The three classic perspective azimuthal projections can show no more than one hemisphere at a time; others (like the azimuthal equidistant) are defined by arbitrary constraints instead of purely geometrical models.

Some projections are inspired by azimuthal principles or modifications of mentioned projections; the result is not, usually, entirely azimuthal itself. E.g., most star-like projections are based on an azimuthal hemisphere.

**Aitoff Projection**
In 1889, David Aitoff announced a very simple modification of the equatorial aspect of an azimuthal equidistant map. Doubling longitudinal values enabled the whole world to fit in the inner disc of the map; the horizontal scale was then doubled, creating a 2:1 ellipse. As a result, the map is neither azimuthal nor equidistant, except along the Equator and central meridian. Neither it is equivalent or conformal.

The Aitoff projection is a very interesting compromise between shape and scale distortion. It clearly suggests the Earth’s shape with less polar shearing than Mollweide’s elliptical projection. However, this influential design was quickly superseded by Hammer's work.

**Hammer Projection**

Properly crediting Aitoff's previous work, in 1892 Ernst Hammer applied exactly the same principle to Lambert's azimuthal equal-area projection.

The resulting 2:1 elliptical equal-area design, called by the author Aitoff-Hammer, by others at first Hammer-Aitoff and then simply the Hammer projection, soon became popular and is used even today for world maps. It was itself the base for several modified projections, like the oblique contribution by Briesemeister.
The strong superficial resemblance of Aitoff's and Hammer's projections led to considerable confusion, even in technical literature.

**Wagner IX Projection**

![Wagner IX map](image)

Part of a series by Karlheinz (Karl Heinrich) Wagner, his ninth proposal (1949) is a rescaling of Aitoff's projection. Parallels are projected as in Aitoff's, but at $7/9$ of their actual value; as a result, the poles are mapped as curved lines along the parallels $70^\circ$N and $70^\circ$S of Aitoff's projection. Therefore, polar angular distortion is lesser than usual in pseudocylindrical projections with polelines. Conversely, meridians are mapped at $5/18$ of the actual value. The projected coordinates are then stretched horizontally and vertically at the reciprocal rates, thus keeping the original aspect ratio (Equator twice as long as central meridian).

The projection is neither equal-area nor conformal. Scale is constant and the same along only the Equator and central meridian.

**Eckert-Greifendorff Projection**

![Eckert-Greifendorff map](image)

Much like Hammer's projection horizontally stretched part of an equatorial equal-area azimuthal map, the projection announced in 1935 by Max Eckert-Greifendorff stretched the corresponding portion of a Hammer
map. In other words, exactly the same idea as Hammer's, but with longitude compressed four times and horizontal scale multiplied fourfold. Before rescaling it uses only a narrow region near the central meridian of the original azimuthal map; as a consequence, parallels are almost straight lines.

**Winkel Tripel Projection**

![Winkel Tripel map](image)

Winkel Tripel map, 50°28"N/S are reference, but not standard, parallels

The third and best known of Oswald Winkel's hybrid projections was called *tripel* (from the German for *triple*). Like his two other proposals published in 1921, it is defined by a simple arithmetic mean including the equidistant cylindrical projection, using an arbitrary value for standard parallels (the author preferred approximately 50°28"N/S; another common value is 40°N/S); these are not standard in the final result. However, the other projection is Aitoff's, therefore, the result is not pseudocylindrical.

Winkel's Tripel projection is peculiarly irregular: it is neither equal-area nor conformal; parallels are straight at Equator and poles, curved elsewhere; scales are constant (but not equal) only at the Equator and central meridian.

Nevertheless, it manages to present a pleasant and balanced view of the world, which led to its choice by several popular atlases. In 1998, it was selected by the prestigious National Geographic Society for its new reference world map, in place of the Robinson projection.

**Conformal Projections**

**Introduction**

A map projection faithfully reproducing all features of the original sphere would be perfectly equidistant; i.e., distances between every two points would keep the same ratio on both map and sphere. Therefore, all shapes
would also be preserved. On a flat map this property is simply not possible (as proved by points at the map's edge).

For many mapping applications (like topography and certain kinds of navigation), a lesser constraint - fidelity of shape, or conformality, is the most fundamental requisite: the angle between any two lines on the sphere must be the same between their projected counterparts on the map; in particular, each parallel must cross every meridian at right angles. Also, scale at any point must be the same in all directions. Conformality is a strictly local property: angles (therefore shapes) are not expected to be preserved much beyond the intersection point; in fact, straight lines on the sphere are usually curved in the plane, and vice versa.

Conformal map projections are frequently employed in large-scale applications, and seldom used for continental or world maps (those shown here are included for comparison only). Since no conformal map can be equal-area (most in fact grossly distort dimensions far from the center of the map), conformal projections are not frequently applied to statistical mapping, where comparisons based on size are common.

Systematic understanding of requisites and properties of conformality had to wait for the development of sophisticated mathematical tools, like differential calculus and complex analysis, in the 18th and 19th centuries. Conversely, conformal mapping became an important branch of modern mathematics.

"Classic" Conformal Projections
For each of the three major projection groups, there is a single conformal design, better presented elsewhere:

- the azimuthal stereographic, which has the unique property of preserving the shape of any circle on the sphere
- the Mercator, a cylindrical projection which, in the normal aspect has straight vertical meridians, therefore enabling direct bearing measurements
- Lambert's conformal conic, a general case of the other two projections

Like most conformal projections, those three suffer from singularities, which are points either

- projected at infinity, so cannot be included in the map, or
- where conformality is absent
In particular (descriptions hold for the normal aspects), the azimuthal stereographic cannot include the point antipodal to the center of projection; Mercator's projection excludes both poles, and the conformal conic shows a single pole, which is nonconformal (since the sum of angles of all meridians is less than 360°).

The "Lagrange" Projection
In the same work (1772) presenting the conic conformal, Johann Lambert included another conformal design. Its development is relatively simple but very interesting:

1. on the sphere, reduce meridian spacing by a factor $n$
2. still on the sphere, change parallel spacing in order to restore conformality to the compressed surface
3. apply an azimuthal stereographic projection in the equatorial aspect

Fortunately, the result of successive conformal transformations is itself conformal.

Even though Lambert developed equations for general $n$ and central parallel, the common equatorial case for $n = 0.5$ is better known as the "Lagrange" projection. Another accomplished mathematician, Joseph Lagrange further generalized Lambert's idea for the ellipsoid.

A "Lagrange" map can show the whole world in a circle. Also, as a consequence of the stereographic step, all meridians and parallels are circular arcs (the central meridian and central parallel are straight lines).
Scale is extremely exaggerated near the poles; conformality also fails at these two points.

This projection is seldom used for actual maps. However, it is the base for many designs, because the sphere mapped on a circle is a fundamental step for conformal mapping.

**Conformal Projections by Eisenlohr and August**

Avoiding singularities was a requisite of two superficially very similar designs from Germany. Both were developed for the equatorial aspect; the Equator and central meridian are straight lines, and the poles are prominent cusps. Scale distortion is strong near the boundary meridians, but both projections are conformal at every point, even the poles.
The design published by Friedrich Eisenlohr in 1870 has two additional features: the scale is constant along the boundary meridians; more remarkably, scale range is the narrowest of any conformal projection: 1 to $3 + 8^{1/2}$. Relatively complex calculations initially restricted its use.

The projection designed by Friedrich August and co-developed by Bellermann was published in 1874 as an alternative to Eisenlohr's design: scale range is wider and not constant at the boundary meridians, but construction is somewhat simpler. A world map is bounded by an epicycloid (the shape defined by a point on a circle rolling without sliding around another, fixed, circle).

Neither Eisenlohr's nor August's projections should be confused with other nonconformal, similar-looking designs like the American polyconic, rectangular polyconic and van der Grinten's IV.

Conformal Hemispheres in Squares
The maturation of complex analysis led to general techniques for conformal mapping, where points of a flat surface are handled as numbers on the complex plane. In particular, three notable cartographers developed aspects of a conformal projection of one hemisphere (or the
whole world, after a suitable rearrangement) on a square. All three approaches require evaluation of elliptic integrals of the first kind.

**Peirce's Quincuncial Projection**

While working at the U.S. Coast and Geodetic Survey, the American philosopher (actually polymath) Charles Sanders Peirce disclosed his projection in 1879. In the normal aspect, it presents the northern hemisphere in a square; the other hemisphere is split into four triangles symmetrically surrounding the first one, akin to star-like projections. In effect, the whole map is a square, inspiring Peirce to call his projection *quincuncial*, after the arrangement of five items in a cross.

Peirce's projection is conformal everywhere except at the corners of the inner hemisphere (thus the midpoints of edges in the whole map), where the Equator breaks abruptly. Scale is highly stretched near those four points; conversely, polar regions are rather compressed. The Equator and four meridians are straight but broken lines; all other graticule lines are complex curves.

Pieces of a quincuncial map can evidently be rearranged as a 2:1 rectangle. Also, the map *tessellates* the plane; i.e., with a trivial rotation, repeated copies can completely cover (*tile*) an arbitrary area, each copy's features exactly matching those of its neighbors.
Guyou's Projection

Only a few years after Peirce, Émile Guyou from France presented his conformal projection (1886-1887). In its original form, it comprises the western and eastern hemispheres, each in a square; the Equator and four meridians are straight lines, two of the later broken along the squares' edges. Other meridians and parallels are complex curves.
Guyou map, central meridian 20°E

Again, scale distortion is great and conformality is absent at the corners of each square (where parallels 45°N/S meet the straight meridians).

Actually, Peirce's and Guyou's projections are transverse cases of each other, emphasizing polar and equatorial aspects, respectively. Guyou maps can also tile the plane.

Hemispheres by Adams
Yet another development of the square theme is an oblique aspect where the poles are placed at two of the square corners. Oscar S. Adams, also a prolific member of the U.S. Coast and Geodetic Survey, presented his conformal world map in two square hemispheres in 1925.

Exactly as in the other aspects by Peirce and Guyou, at the square's corners scale distortion is extreme and the map is not conformal. Only the Equator and the central meridian are straight lines; the boundary meridians are also straight but broken at the Equator.

Despite interest due to their mathematical development, the conformal projections in squares of Peirce, Guyou and Adams were seldom used.

**Adams's world in a square (1929)**

**Conformal World Maps in Other Polygons**

**New Complex Tools**

Further development on conformal projections mainly relied on notable results of complex analysis:
• Riemann's theorem on conformal mapping (1851), which states conditions necessary for conformally converting between two connected plane regions, but does not describe how to achieve this mapping
• Hermann A. Schwarz's integral on the complex plane for mapping a circle with radius 1 (called the *unit disk*) to any regular polygon
• the Schwarz-Christoffel transformation, another complex integral independently demonstrated by Elwin B. Christoffel in 1867 and Schwarz in 1869; it expresses how to map between half a plane (or the unit disk) and *any* simply connected (i.e., not self-intersecting) polygon.

![Adams's world in a square (1936)]

Although the works of Schwarz and Christoffel realized a constructive proof of Riemann's theorem, their application to cartography (other than simple, particular instances) remained impractical for nearly one century; for most cases, they do not result in closed formulas and require solving a system of nonlinear equations. Actual mapping involves lengthy numerical evaluation by successive approximations.

Even after digital computers became generally available, results were far from uniform. Many algorithms for Schwarz-Christoffel mapping suffered from low efficiency, limited precision, or instability, i.e., failure to converge to a result, or poor handling of singularities (usually present at polygon vertices).
World Maps by Adams

After presenting his conformal hemispheres in squares, O.S. Adams proposed two projections with a world map in a single square.

The first one (1929) has poles in opposite corners; scale distortion is extreme at each corner, which lacks conformality. The second version (1936) has poles at midpoints of opposite edges. Again, there's strong scale distortion at the vertices. This projection is not conformal at each corner and the two poles.

Other less-known conformal projections by Adams were based on an ellipse and several other polygons.

Lee's Conformal Maps

Laurence P. Lee, distinguished cartographer and senior officer at a national mapping agency in New Zealand, further generalized and improved the accuracy of methods for arbitrary conformal mapping. His projections included maps of the world on an equilateral triangle and on the faces of regular polyhedra. The well-known triangular projection (1965) is conformal everywhere except at the vertices.

Like Adams's, Lee's designs attracted academic interest and paved the way for new mathematical achievements, but found limited usage in common maps.
Newer Conformal Maps

Constant Xarax from Greece, influenced by Lee's conformal maps on polygons, Briesemeister's oblique projection and polyhedral maps in a butterfly arrangement, proposed a conformal map of the world in half a regular hexagon (2004). Essentially a three-lobed design, the result balances legibility, low interruption count and easily recognizable shapes.

Other Interesting Projections

Countless projections were devised in centuries of map-making. Many designs cannot be readily classified in the main groups (azimuthal, cylindrical, pseudocylindrical, conic or pseudoconic), even though their design is similar or derived.

A large number of projections whose graticule lines are circles or derived conic curves with different radii and centers are called polyconic (not to be confused with the particular group of polyconic projections). This is a broad and artificial category comprising otherwise unrelated projections.

Projections by Van der Grinten
An American, Alphons J. van der Grinten published in 1904 and 1905 two projections, the first one devised as early as 1898. Both were designed for the equatorial aspect, with straight Equator and central meridian; all other parallels and meridians were circular arcs, with nonconcentric meridians regularly spaced along the Equator.

Bludau proposed two modifications to the first version; the four designs soon came to be collectively (and confusingly) called "van der Grinten" projections:

I. the first original projection, bounded by a circle
II. Bludau's modification of I, with parallels crossing meridians at right angles
III. Bludau's modification of I, with straight, horizontal parallels
IV. the second original projection, bounded by two identical circles with centers spaced 1.2 radii apart

Van der Grinten's proposals are examples of conventional designs, derived not from a perspective process but from an arbitrary geometric construction on the map plane. They are neither equal-area nor conformal (despite a superficial resemblance to projections by Lagrange, Eisenlohr and August), but intended to "look right", in the sense of conveying the notion of a round Earth without departing too much from Mercator's familiar shapes.
Van der Grinten's second projection (IV)

The best known of all four, van der Grinten's I, also known simply as the Grinten projection, was widely used, especially after its choice for reference world maps by the National Geographic Society from 1922 to 1988. Of the others, only the III variant saw limited use. Although the poles can be included in the map, areal distortion is large at high latitudes, thus most van der Grinten maps are clipped near parallels 80°N and 80°S.

Globular Projections by Maurer

Maurer's "full-globular" map

The ancient group of globular projections includes circular arcs for both meridians and parallels, and maps ordinarily limited to a single hemisphere. H. Maurer presented in 1922 three conventional projections resembling globular features. The "full-globular" projection has meridians spaced like in van der Grinten's IV projection; parallels are equally spaced along the boundary meridians, and both the central meridian and the Equator have constant scale. Each boundary meridian spans half the limiting circle, thus the whole world is set resembling a double-edged ax.
His two other globular proposals are called "all-globular" and "apparent-globular".

**Orthoapsidal Projections**

Beginning in 1943, the notable cartography teacher and author Erwin Raisz created a series of orthoapsidal projections mapping the sphere onto intermediary surfaces. However, instead of "unrolled" like in cylindrical or conic maps, each surface is then projected orthographically onto the final plane.

![Orthoapsidal map](https://example.com/orthoapsidal.png)

Orthoapsidal ("Armadillo") map on part of a toroidal surface; tilt angle 20°, central meridian 10°E. Raisz's original map extended the eastern and western edges, with parallels spanning about 410° in order to avoid splitting Alaska and Siberia.

In the best-known orthoapsidal projection, called Armadillo (since it vaguely resembles the curling armored mammal), the sphere is mapped onto 1/4 of a degenerate torus with radii 1 and 1, which resembles a doughnut with a zero-sized hole. Parallels and meridians are equidistant circular arcs on the torus, but nonequidistant elliptical arcs in the final map.

![Development of the Armadillo projection](https://example.com/development.png)

Development of the Armadillo projection: the sphere is mapped to the region resembling half of a car tire, and that region to the blue projection plane.

In the conventional form of the Armadillo map, Raisz preferred 10°E as the central meridian; the torus is then tilted 20 degrees and orthographically flattened onto the projection plane. Southern regions like
Patagonia, New Zealand and Antarctica are hidden from view, and sometimes presented separately.

Orthoapsidal map on a half-ellipsoid, eccentricity 1.75, tilt angle 20°; central meridian 10°E

Raisz also developed a map on one half of an oblate ellipsoid of rotation; the intermediate process is roughly a three-dimensional analogue of that applied by Aitoff to the azimuthal equidistant projection.

Another surface employed by Raisz was one half of a tilted hyperboloid of rotation of two sheets; in this case, a North polar map was interrupted in four identical lobes, resembling Maurer's S231 projection and, different from other orthoapsidal designs, showing the whole world. As drawn by Richard Edes Harrison, this projection was prominently featured in the cover of Scientific American 233(5); it is interrupted (at 60°E, 150°E, 120°W and 30°W) south of, apparently, 10°N. Harrison, known for his innovative and detailed maps, is quoted as characterizing it as "the most elegant of all world maps".

Orthoapsidal maps are neither conformal nor equal-area; parallels and meridians do not necessarily hold properties (like equidistance) of the intermediary surface.

**Projections by Arden-Close**
Charles F. Arden-Close designed some map projections by averaging; his best-known (1943) is a simple arithmetical mean of one hemisphere of an equatorial equal-area cylindrical map with its transverse aspect, the Equator in one map coinciding with the central meridian in the other. Shaped like a square with circular corners, the result is neither conformal nor equal-area. Doubling coordinate values, the method can be easily extended in order to show the whole world.

**Oblique Projections**

Hammer map with coordinates rotated 45° counterclockwise. Every piece on Earth is represented (Asia was not cut off, but cut up to the middle and stretched out).

Most maps published in atlases are oriented with the North Pole at top, South Pole at bottom and the Atlantic Ocean somewhere in the middle. Supposing a perfectly spherical Earth, such setting is just a convention - one could first rotate the globe in any way and afterwards project the rotated coordinates as usual. A Middle-Age world map drawn by Western ambassadors to China, surely due to political reasons, put the Northern Hemisphere at the bottom and China nearer the center than usual.
If neither Equator nor the central meridian are aligned with and centered on the map axes, the result is commonly called an *oblique* projection (or, more properly, an oblique *map*). Although general properties of the original projection (like area and shape equivalence) still hold, those depending on the graticule orientation are generally not preserved.

The Atlantis map was named after an ocean, not a myth.
A common reason for tilting a projection is moving a large, important area to the places of lesser distortion. The *Atlantis* map (Bartholomew, 1948) presents the Atlantic Ocean in a long, continuous strip aligned with the map's major dimension. Also clearly showing the Arctic "ocean" as a rather small extension of the larger Atlantic, it is an oblique Mollweide projection centered at 30°W, 45°N.

Two other maps emphasizing sea regions were announced by Athelstan Spilhaus in 1942, one using Hammer's and the other August's conformal projection, both with 15°E, 70°S as the map center: very few oceanic sites (notably the Caribbean sea) are interrupted, and relative sizes of oceans are clearly expressed. With modern computers, finding the appropriate rotation parameters for such a "good" distribution of features is fairly easy; one can only imagine the laborious process employed by the original author. Much later, Spilhaus extended his ideas using interrupted maps.

Of course, sometimes a fundamental requirement (like keeping coordinate lines straight or parallel, or preserving correct directions along the meridians in azimuthal projections) prevents adoption of oblique maps.
A fact often overlooked is that points at the borders of any world map are represented at least twice, since in the original sphere the "edges" are joined (an unrelated phenomenon occurs in cylindrical and other flat-polar projections like Eckert's and mine, which stretch the poles into line segments). We are used to that obvious scale distortion (points in a neighborhood get widely separated in the map), so we hardly notice it in conventional maps, save maybe at the extreme tips of Siberia and Alaska.

Hammer map centered on Eurasia, with a double pole

Oblique maps often make this kind of distortion obvious: notice the New Zealand islands in the Atlantis map, and the east and west coasts of North America in Spilhaus's maps. Rotating a projection in order to put a temperate region at the center can easily create a two-pole map. Although one could probably use a polyconic projection for this purpose, the map above nicely shows Eurasia with a shape nearer to reality than usual, at expense of North America and Antarctica and, as shown by the projected graticule, dramatic angular deformation at the southern Indian Ocean.

Briesemeister map, a simple modification of Hammer's projection clearly presenting all land masses except Antarctica, with a double pole

A similar, very simple modification of Hammer's projection was published by William Briesemeister in 1948: the map is first projected obliquely with 10°E 45°N as the central point; then it is linearly stretched to an aspect ratio of 7 : 4 (compare with 2 : 1 for both Mollweide and conventional...
Hammer). As a result, parallels in the vicinity of the North Pole are almost circular.

Still another oblique version of Hammer's method is the *Nordic* projection (Bartholomew 1950), centered at 0°W, 45°N: it closely resembles Briesemeister's map, without the rescaling.

![An equidistant cylindrical map with Campinas, Brazil infinitely stretched on the bottom edge; its antipode lies at the top edge](image)

While the *azimuthal equidistant* projection preserves distances radially from the center, the *equidistant cylindrical* projection preserves distances vertically. In particular, distances of any point to the projected "poles" can be directly read as the y-coordinate from the top or bottom of the map. The bearing from either projected "pole" is also immediately available as the x-coordinate from the center, since meridian spacing is uniform in the original map. Thus, this map has the same purpose (actually it is also appropriate for the *antipodal* point) as the familiar *azimuthal* map, but shows the bearing without a protractor.

**Interrupted Maps**

**Introduction**

![Globe wrapped in interrupted sinusoidal map](image)
No map projection can preserve shape and size simultaneously, and the larger the mapped area, the more pronounced the total distortion. **Rectangular** world maps are prone to excessive area and distance stretching, while those using circular and elliptical projections usually present too much shape distortion at the periphery.

**Interrupted** maps attempt a compromise, "cutting" the terrestrial surface along some arbitrarily chosen lines and projecting each section, or **lobe** (or **gore**, in case interruptions repeat periodically along meridians), separately with lower stretching. Commonly lobe boundaries are designed to fall upon less important (for the map's purpose) areas, like oceans.

In a sense, interrupting a map creates another kind of distance **distortion**, since neighbor points on the sphere become widely spaced in the map; therefore, too many lobes negate the benefits of interruption. As mentioned for **oblique** maps, such distortion actually happens at the edges of any ordinary projection.

Interrupted projections were used by Waldseemüller (1507) and Leonardo da Vinci (ca. 1514), among others. An early example was a variation of **Werner's projection** by Mercator (1538).

### Simple Interrupted Maps

The **sinusoidal (also known as Sanson-Flamsteed)** projection has a **simple construction** and interesting features: **pseudocylindrical**, **equal-area** and constant vertical **scale**. On a whole-world equatorial sinusoidal map, the polar regions at extreme longitudes suffer from strong shape distortion (shearing). Interrupting the map preserves its better features with lesser shearing.
Interrupted sinusoidal map, each hemisphere split in nine lobes

Clearly there is a trade-off: increasing the number of lobes further reduces shape distortion as each lobe is centered around its own different meridian, until the discontinuities make the map more a curiosity than something useful in its planar form. However, a lobed map could, if printed on a sheet of flexible material, cut and joined at the borders, make up a fairly good globe (interestingly enough, ancient gore maps had exactly that purpose, albeit with a different lobe arrangement and much more primitive projection methods).

Another gore map. Since it is based on the polyconic projection, parallels are curved and it is not equal-area. Global areal distortion is not as pronounced as in corresponding conterminous map.

Maps with lobes in a row along the Equator make clear why cylindrical projections necessarily distort polar regions: they must horizontally stretch and fasten them together in order to force a rectangular map.
Finally, as usual designing a map reflects the author's particular point of view. An asymmetrical arrangement of lobe boundaries can avoid cutting the three major oceans instead of land masses. In the case of a sinusoidal projection, all other properties still hold, including mapped area. Asymmetrical lobes are featured in classic interrupted maps by Goode, Boggs and McBryde.

After using Hammer and August projections for designing maps emphasizing near-continuous seas, Spilhaus proposed more complicated approaches, like a modified (not equal-area) Hammer map interrupted in three lobes, or interrupting the map at shorelines instead of, as usual, graticule lines.

**Star Projections**

**Introduction**

Since most continental area is concentrated around the North Pole, some star-like interrupted projections were designed having part or all of the northern hemisphere as a more or less circular shape and splitting the remaining surface in lobes placed around it instead of in a row. Of course, for ordinary maps only the north polar aspect is interesting and Antarctica is always divided. A few projections emphasize oceanic areas by inverting this pattern.

Most star maps are actually composites, employing azimuthal or near-azimuthal projections for the central "core", which is bounded by an arbitrary parallel. The lobes are projected differently but frequently retain the core's parallel spacing.

**Early Maps: Projections by Jäger and Petermann**
Star-shaped maps probably date from works by Leonardo da Vinci (1514) and Guillaume Le Testu (1556).

One of the earliest modern designs was the compromise projection published by G. Jäger (1865). In its polar aspect, all meridians in the core hemisphere are straight lines evenly spaced, but with different scale; the Equator and eight arbitrary meridians define eight triangular sectors. Parallels in this hemisphere are straight lines broken at sector boundaries, perpendicular to and equally spaced along each sector's central meridian. The outer hemisphere comprises eight triangles mirroring the core sectors, therefore with simple straight lines for parallels.

Since the meridians bounding sectors are not evenly spaced, the core octagonal hemisphere is irregular, and lobes differ in width and length; later star projections usually have symmetric lobes.

Petermann almost immediately (1865) reproduced Jäger's proposal but greatly modified the map with distinct projection methods for the core and outer hemisphere:

- the core hemisphere uses an ordinary **azimuthal equidistant** projection
- the outer hemisphere is divided in eight identical lobes (although I have found conflicting descriptions)
- in each lobe, the central meridian is a straight line; all other meridians are straight lines broken at the Equator
- in each lobe, parallels are concentric arcs of circle spaced exactly like in the core hemisphere

Almost unused, Jäger's and Petermann's projections are neither conformal nor equal-area.
Berghaus's Star Projection

This attractively shaped projection was created by the German Hermann Berghaus in 1879. It is actually a special case of Petermann's projection with five lobes. Consequently, all parallels are concentric, equally spaced arcs of circle, and meridians are straight lines, all but five broken at the Equator. Also, the projection is of course azimuthal in the northern hemisphere, but neither conformal nor equal-area.

Although other arrangements are possible, Berghaus recommended interrupting the southern hemisphere at 56°E, 128°E, 160°W, 88°W and 16°W. As a result, of all major land masses only Australia and Antarctica are divided.
Out of curiosity, I used Berghaus's approach with other numbers of lobes, although less than five is prone to greater distortion and much more causes excessive land cutting, thus defeating the projection's purpose.

Several star maps can be cut, folded and assembled into pyramids; the 3-point Berghaus map is an equilateral triangle foldable into a regular tetrahedron.

**Conoalactic Projection**

![Conoalactic map, interrupted in South hemisphere at 90°W, 0°, 90°E](image)

Introduced by Steinhauser in 1883, the conoalactic projection is actually very similar to Petermann's and Berghaus's in construction. The northern hemisphere is based on a simple equidistant conic projection, while the southern hemisphere is split in four lobes having equidistant parallels and straight meridians broken at the Equator. It is neither equal-area nor conformal; scale is constant along the standard parallels 0°N and 4°17′53″N and in the four unbroken meridians. Other meridians preserve distances only in the northern hemisphere.

Notwithstanding its unusual shape, this projection can also be classified as star-like.

**Maurer's Star Projections**
Hans Maurer developed a monumental effort to catalogue and organize every possible projection according to hierarchical categories and rigidly defined criteria (1935). He also designed many new projections, sometimes for illustrating gaps in his classification system.

The two projections Maurer called S231 and S233 are star-shaped. The first extends the northern hemisphere of a Lambert azimuthal map and is also equal-area, even in the interrupted hemisphere. The meridians in each lobe are not straight but delicately curved lines. Parallels are concentric arcs of circle symmetrically spaced around the Equator. Meridian spacing along parallels is also symmetric above and below the Equator.
Maurer's S233 projection, interrupted from 20°W.

Despite a superficial resemblance to a 6-pointed Berghaus map, Maurer's S233 is even simpler: also centered on a pole, all coordinate lines are straight lines. Parallels are equally spaced everywhere, broken at the interruption meridians. Each lobe is symmetrical above and below the Equator. The result is neither conformal nor equal-area. It is essentially a symmetric modification of Jäger's projection.

Although described with six lobes, both star projections can be easily generalized to any number, at least two (S231) or three (S233).

"Tetrahedral" Projection
John Bartholomew, fourth of a long lineage of mapmakers holding this name and surname, designed several projections based on previous works; some like the Atlantis are oblique aspects, while others are composite. Among the latter group, one (1942) combined part of an azimuthal equidistant hemisphere (bounded by the 23°30' parallel) with three identical lobes based on Werner's projection extending to the opposite pole. The three straight meridians have constant scale. Unfortunately, only at the poles the parallel scale is the same in the two projections, therefore meridian spacing must be stretched in the lobes by approximately 26.6%.

Although it has no relation to true polyhedral maps, this projection is called "tetrahedral", maybe due to a vague resemblance to the face arrangement of a tetrahedron's fold-out. Neither equal-area nor conformal, it was used in both North and South polar aspects.

Classic Interrupted Maps

Interrupted Mollweide Map
Interrupted Mollweide hemispheres, central meridians 110°W and 70°E

Pseudocylindrical projections are especially appropriate for linear interruptions along meridians, like in this divided Mollweide map: meridians are still mapped to elliptical arcs, and those at (after any oblique rotation) 90°W and 90°E are circular. This interrupted form with symmetrical central meridians comprises two perfect circles. Compare the azimuthal orthographic and stereographic maps of exactly the same regions.

Interrupted Mollweide map, simplified lobes

In 1916, before designing his most famous map, J.P. Goode experimented interrupting a pure Mollweide projection. In an arrangement slightly more complex than that shown here, the result was popular but eventually superseded by the true homolosine projection.

Interrupted Goode Homolosine Projection
Interrupted Goode homolosine map; Iceland and portions of Greenland and Eastern Asia appear twice.

This is a common form of J.P. Goode's homolosine projection, easily recognized due to its broken meridians. The lobe arrangement shown here is similar to that originally published by Goode (1923-1925). Some maps of this kind include extensions repeating a few portions in order to show Greenland and eastern Russia uninterrupted.

Interrupted Boggs Eumorphic

Interrupted eumorphic map with extensions repeating Greenland and the Bering sea region. The break in Eurasian meridians is clearly visible.

Boggs preferred his hybrid eumorphic projection in an interrupted form resembling Goode's homolosine maps. Since it averages the sinusoidal and Mollweide projections instead of joining bands projected separately, meridians in an eumorphic map are unbroken except (arbitrarily) at the Eurasian lobe: Boggs usually employed a different central meridian north of the 40°N parallel.

Interruption Devices

The main purpose of interrupting a map is moving significant regions to less-distorted places, usually near the center of each zone. Several cartographic tricks enhance its usefulness.

Recentering and Cropping
Interrupted Eckert IV map with lobes recentered: only one meridian (not necessarily the middle one) in each lobe is mapped to a straight line.

Especially with pseudocylindrical projections, each lobe of an interrupted map may easily be projected with its own, arbitrary central meridian, not necessarily the same as the median one. This introduces asymmetrical angular distortion, privileging regions near the straight central meridian while detracting from others. The central meridian may even change with latitude, like in the Eurasian lobe of Boggs’s eumorphic map.

Recentering is also an effective device for uninterrupted continental or regional maps. The region of interest is centered, minimizing distortion, and the remaining projected area is cropped off.

Ordinary map
Meridians recentered
Fully recentered

Three aspects of the Japanese islands using the same Eckert IV projection. Every regional map covers the same area but slightly different regions. The second option is better for this particular projection, whose least-distortion parallel is not the Equator, but instead near the interest area. The oblique map is actually more distorted due to exaggerated latitudinal scale close to the center.

Condensing and Insets
Some cartographic devices are actually editorial tricks, intended for clarity or printing convenience. They do not affect distortion patterns, and can be employed purely as lay-out tools.

Frequently applied simultaneously with interruption, **condensing** the map means removing unimportant areas and joining the remaining sections. It can save publishing space or, conversely, allow larger scales and better detail in the same printed area.

Also an editorial tool, an **inset** is a small illustration detached or superimposed on the main map, useful:

- for presenting an area strongly distorted or interrupted, or impossible to be shown (like in **Mercator's projection**) by the main projection; the **azimuthal equidistant** is commonly employed for polar regions
especially for large-scale maps of a small or lesser-known region, for quickly pinpointing its location on Earth; the azimuthal orthographic is frequently used in small hemispheres

**Abusing Interruptions**

Like several cartographic techniques, interruption can be misused or thought of as purely editorial convenience. Hastily or carelessly prepared maps may suffer from:

- failure to clearly mark condensed areas
- interruption gaps not marked, suggesting continuity instead

Such maps may be good enough for advertisement, but unacceptable for didactic or scientific purposes.

Interrupted Eckert IV map, not condensed but with lobe gaps colored like additional sea area; partial graticule and removal of Antarctica help hiding the flaw. Although continental shapes are better presented than in rectangular maps created by cylindrical projections, land/water area ratios are misleading and distance between, e.g., Iceland and Greenland is greatly stretched.

**Polyhedral Maps**
Most cartographic problems would disappear if the Earth were a polyhedron

**Introduction**

Several approaches were presented for reducing distortion when transforming a spherical surface into a flat map, including:

- first mapping the sphere into an intermediate zero-Gaussian curvature surface like a cylinder or a cone, then converting the surface into a plane
- partially cutting the sphere and separately projecting each division in an interrupted map

Both techniques are combined in *polyhedral* maps:

1. inscribe the sphere in a polyhedron, then separately project regions of the sphere onto each polyhedral face
2. optionally, cut and disassemble the polyhedron into a flat map, called a "net" or fold-out

Intuitively, distortion in polyhedral maps is greater near vertices and edges, where the polyhedron is farther from the inscribed sphere; also, increasing the number of faces is likely to reduce distortion (after all, a sphere is equivalent to a polyhedron with infinitely many faces). However, too many faces create additional gaps and direction changes in the unfolded map, greatly reducing its usefulness.

Polyhedral maps are completely unrelated to "polyhedicric" projections, used in several variants circa 1900 for large-scale mapping.
**Common Polyhedra**

If the polyhedral faces cover (i.e. *tile* or *tessellate*) the plane when juxtaposed, the map can be useful even in its unfolded form. Any triangle or quadrilateral tiles the plane, like a regular hexagon does, but the regular pentagon does not.

The five regular or *Platonic* polyhedra (whose faces are identical regular polygons, and with identical angles at each corner) are natural candidates for polyhedral maps, although distortion is usually unacceptable in the tetrahedron.

<table>
<thead>
<tr>
<th>Solid</th>
<th>Common names</th>
<th>Faces (all regular)</th>
<th>Face edges/vertex</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Regular tetrahedron" /></td>
<td>Regular tetrahedron, regular triangular pyramid</td>
<td>4 triangles</td>
<td>(3 x) 3</td>
</tr>
<tr>
<td><img src="image" alt="Regular hexahedron" /></td>
<td>Regular hexahedron, cube</td>
<td>6 squares</td>
<td>(3 x) 4</td>
</tr>
<tr>
<td><img src="image" alt="Regular octahedron" /></td>
<td>Regular octahedron</td>
<td>8 triangles</td>
<td>(4 x) 3</td>
</tr>
<tr>
<td><img src="image" alt="Regular dodecahedron" /></td>
<td>Regular dodecahedron</td>
<td>12 pentagons</td>
<td>(3 x) 5</td>
</tr>
<tr>
<td><img src="image" alt="Regular icosahedron" /></td>
<td>Regular icosahedron</td>
<td>20 triangles</td>
<td>(5 x) 3</td>
</tr>
<tr>
<td><img src="image" alt="Truncated octahedron" /></td>
<td>Truncated octahedron</td>
<td>8 hexagons, 6 squares</td>
<td>4, 6, 6</td>
</tr>
<tr>
<td><img src="image" alt="Cuboctahedron" /></td>
<td>Cuboctahedron</td>
<td>6 squares, 8 triangles</td>
<td>3, 4, 3, 4</td>
</tr>
<tr>
<td><img src="image" alt="Rhombicuboctahedron" /></td>
<td>(Small) rhombicuboctahedron</td>
<td>18 squares, 8 triangles</td>
<td>3, 4, 4, 4</td>
</tr>
<tr>
<td><img src="image" alt="Truncated icosahedron" /></td>
<td>Truncated icosahedron</td>
<td>12 pentagons, 20 hexagons</td>
<td>5, 6, 6</td>
</tr>
</tbody>
</table>

*Basic features of a few regular and semiregular polyhedra; the octahedron, icosahedron and cuboctahedron have been applied to commercial maps, like a different form of truncated octahedron.*
Some semiregular and uniform (whose faces are regular polygons and vertices are congruent) polyhedra have also been considered for projection.

The idea of using solids as maps goes back at least as far as A. Dürer, even though he did not actually design more than fold-out drafts as part of a general treatise on perspective (1525, revised in 1538).

The most frequently used method for projecting faces uses the gnomonic projection for each section, followed by conformal approaches.

The 3-point variant of Berghaus's star map is incidentally foldable as a tetrahedron, although its development is unrelated to any method aforementioned.

Despite the common name, Bartholomew's tetrahedral projection is actually a star-like composite, unrelated to polyhedra.

The concept of truly tetrahedral pseudoworlds was used by the Dutch artist M.C. Escher in his fanciful engravings Double planetoid (1949) and Tetrahedral planetoid (1954). Tetrahedral "globes" suggest a new meaning for Isaiah 11:12 ("He will assemble the scattered people of Judah from the four quarters of the earth").

**Cubic Globes**

![Gnomonic cubic map, graticule spacing 10°, poles in opposite corners; printable versions available](image)

Although mapping into the regular hexahedron (an ordinary cube) is prone to strong distortion, the nonsensical notion of "Earth-in-a-box" has always attracted me. Once I plotted and folded such a map by hand alone. Fortunately now I own a computer...
Different arrangements of six gnomonic square faces were used by Reichard (1803) and other cartographers, mainly for celestial atlases. Like in all gnomonic maps, great circles (including the Equator and meridians) are transformed to straight lines, except where broken at face edges.

**Cahill's Butterfly Map**

Starting in 1909, Bernard Cahill patented several maps based on the octahedron, using gnomonic, conformal or arbitrary projections. All were based on eight equilateral triangles which could be arranged in several ways, the commonest called a "butterfly map". Here it is presented in the gnomonic form with poles in opposite vertices, cutting meridians every 90°. Other variants are conformal or equal-area, but include additional interruptions or slightly curved edges. Apparently no Cahill map was ever much popular, even after thirty years of promotion by the author.

The butterfly lay-out, superficially resembling the conoalactic projection, benefits from the continental distribution much like done in star projections.

**A Modified Collignon Map**
Collignon's curious projection can be modified to a "butterfly" variant in three straightforward steps: interrupting the diamond-shaped version along three meridians, creating eight triangular lobes; changing both horizontal and vertical scales in order to make lobes equilateral while keeping area constant; and rearranging the lobes around the North pole. The second step can be omitted yielding a slightly different map which folds into an elongated irregular octahedron. Either map is still equal-area but, of course, pseudocylindrical only at each lobe.

Mapping to Truncated Octahedra

Using the Semiregular Truncated Octahedron

Projecting the world gnomonically on an octahedron is a fairly simple task since all meridians and the Equator are mapped into straight lines, therefore octant boundaries are easily deduced from map coordinates.

The same projection applied to a truncated octahedron reduces area distortion in part of the map since the six original vertices are clipped, or
more properly "flattened" into square faces closer to the inscribed spherical surface. Splitting each new face into four right triangles introduces very few additional interruptions in land masses compared to the original butterfly map, except in North America and Northern Asia.

**Waterman's Projection System**

Steve Waterman recently devised a polyhedral projection addressing both distortion and partitioning of land masses. It is based on a truncated octahedron, but not the semiregular type whose faces are regular polygons.

Waterman studied *sphere-packing* - the old mathematical problem of juxtaposing identical spheres in the smallest possible volume - and compiled a list of polyhedra whose vertices are defined by the centers of a set of packed spheres. One of those, called W5, is the foundation of this projection. All square faces are split in four right triangles, except the one with the South Pole, which also borrows narrow slices of adjacent hexagonal faces, thus keeping Antarctica whole. The opposite face is split, leaving interruptions in Arctic islands.

Like in other polyhedral maps, the lobes can be rearranged depending on the map's purpose. For instance, the left and right map halves can be joined at the South Atlantic edges, or they can be swapped for a conterminous North or South Pacific Ocean.

In its basic form, this projection is neither conformal nor equal-area. The butterfly lay-out combines legibility and low distortion.
Icosahedral Maps

With the highest face count among regular polyhedra, the icosahedron was long a favorite for maps.

R. Buckminster Fuller (made famous by geodesic domes and other innovative engineering ideas) designed several polyhedral (like other of his creations, formally named Dymaxion™) maps, at first on a cuboctahedron (1943), later adopting the icosahedron. All were patented and heavily promoted; some icosahedral versions further subdivide a few triangular faces, thus almost completely avoiding split land masses. Most Fuller maps employed arbitrary projections, usually with constant scale along face edges.

Another icosahedral map was briefly made popular by Fisher and Miller, ca. 1944.

Maps on a dodecahedron

Perhaps the most globe-like of all five regular solids is the dodecahedron (its volume differs the least from that of a inscribed sphere; on the other hand, the icosahedron has the bigger volume/surface ratio, and its volume
best approximates that of a circumscribed sphere); unfortunately its faces don't tile a plane so most faces in a fold-out would be connected by only one or two edges, causing too many gaps.

**Rhombicuboctahedral Maps**

Rhombicuboctahedral map fold-out, "central" meridian 0°; more faces mean lesser distortion, but also less continuity. [Printable version available](https://example.com).

Two assembled rhombicuboctahedral pseudoglobes, with poles centered on opposite square or triangular faces

In comparison with the previous solids, the rhombicuboctahedron looks pleasantly roundish due to a larger face count. However, its unfolded form makes evident the problem of finding a suitable distribution of features in multiple faces without too many cuts.

**Map Fold-outs**

Print, cut, fold and glue paper polyhedra to create your own pseudoglobe.
Assembling
Some assembly tips:

- in the map fold-outs, folding lines separating polygon faces are absent; they are included in the crease patterns available here for practice
- precision and patience matter most. Small errors can accumulate and prevent fitting the last faces
- using a dull blade (or a spent ballpoint pen) guided by ruler or straightedge, lightly and carefully score all folding lines before cutting. This will make creasing easier and more precise.
- if you prefer scoring the paper's reverse side to avoid scratching the inked surface, use a needle or pin to punch tiny guide holes at every vertex.
- a utility knife guided by a straightedge is more precise than scissors, but please be careful! To keep better alignment, score or cut as many lines as possible without moving the straightedge; also, use the extra alignment ticks included in several maps.
- check every white tab's fit before applying glue.
- a single face (shaded in the preview pattern) should be glued last; it has no tabs, so must be aligned by sight.
- as a rule of thumb, polyhedra with more (and smaller) faces are harder to assemble. If you take the punch-score-cut-fold approach, expect to spend 7-10 minutes on the tetrahedron, and almost two hours on the truncated icosahedron.

Fold-outs

World maps on a regular **tetrahedron** (5 high-resolution maps, 8 low-resolution)

World maps on a cube **(cube)** (10 high-resolution maps, 7 low-resolution)

World maps on a regular **octahedron** (5 high-resolution maps, 5 low-resolution)

World maps on a regular **dodecahedron** (8 new high-resolution maps, 9 low-resolution)

World maps on a regular **icosahedron** (8 high-resolution maps, 12 low-resolution)
Disclaimers

Although I have carefully programmed my mapping algorithms, I offer no warranty about the precision of these images. The data sets used may be outdated, especially those with country borders.

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Premodern Projections

Introduction

Of several projections created before or at the Renaissance, most have fallen in disuse long ago, and only a few with some outstanding properties are remembered today. Much information about these projections is uncertain:

- several were often reinvented by obscure authors
- mostly fragmented, imprecise descriptions survive today, sometimes in indirect references or rough copies
- especially during Middle Ages, most maps intended only illustrating the Earth features, thus lacking precise contours and, frequently, a graticule
due to inaccurate copying or primitive drafting techniques, many maps do not precisely reflect the author's original design; frequently straight or circular lines were used as an approximation to more complex curves.

- as a consequence, it is not easy ascertaining the correct meaning of graticule lines
- if the graticule is absent, measurement and inverse mapping of features have limited value due to poor geographical knowledge at ancient times

Projections mentioned here are of mainly historical significance; some where already presented when fitting one of the main projection groups. A strong Western bias is evident as I have little information on cartographic development in other cultures.

**Equidistant Cylindrical and Trapezoidal Maps**

The equidistant cylindrical projection was very common since before the Christian era, no doubt due to a simple and practical construction.

Maybe as old or older, the trapezoidal projection has similarly horizontal, straight, equally spaced parallels, but the meridians converge, not necessarily at a point. Thus at most two parallels (four in the broken meridian case) and only one meridian are standard lines. It is neither equivalent nor conformal.

Most maps for this primitive and obsolete pseudocylindrical projection show only part of the northern hemisphere; some include the whole Earth with symmetrical hemispheres, while others are truly trapezoidal with meridians unbroken at the Equator.

Depending on the choice of standard parallels, a trapezoidal map may resemble the Collignon, Eckert's I and II projections. It is also a general case of the equidistant cylindrical, for standard parallels with different circumferences.
Azimuthal Projections

It is remarkable that some azimuthal projections, important even today, were known more than two thousand years ago by the Greeks and maybe by the Egyptians. The orthographic, stereographic and gnomonic projections are all based on solid principles of perspective and Euclidean geometry, while the azimuthal equidistant dates from the 15th century and is constructed arbitrarily.

The orthographic projection, one of the most realistic maps in a pictorial sense, provides an interesting contrast to the much later "globular" approaches.

Globular Projections

Apparently, much of the rich scientific inheritance from Antiquity was forgotten or ignored during most of the Western Middle Ages; however, despite the widespread modern myth, the idea of a spherical globe was never banned on religious terms and was accepted by most learned people (Columbus argued not about a round, but, mistakenly, a small Earth).

The so-called globular maps were essentially simple pictorial devices for presenting general geographic features. Their main purpose was emphasizing the Earth's roundness; no globular projection is equal-area or conformal. Originally, all were restricted to one hemisphere bounded by a circle, with only equatorial aspects considered. Both the central meridian and Equator are straight, perpendicular lines. Basic geometrical constraints, summarized below, define all historic designs.

In spite of superficial similarity to azimuthal projections, globular maps are not developed by proper perspective rules: the graticule is arbitrarily placed using easily drawn curves. As usual in cartography, no approach is perfect: although Fournier's second work best matches the visual aspect of a three-dimensional globe, the "Nicolosi" map has possibly the least global distortion of shape; latest to be widely published, the latter was almost certainly the very first to be invented.
One of the oldest known arbitrary projections in a hemisphere was described by the great philosopher Roger Bacon (ca. 1265) and survived due to works by Monachus (ca. 1527) and d'Ailly. The design was modified in two proposals by Apian (1524), one of which was used by Tramezzino (1554) and extended by Agnese and Ortelius.

Fournier introduced two modifications to the globular style in 1643, the first using circular arcs instead of straight lines as parallels. A further change was popularized by Giovanni Nicolosi's projection of 1660 which, although also attributed to La Hire in 1794, was probably devised by al-Biruni, ca. 1000. Most modern mentions to "globular" maps refer to the "Nicolosi" design, widely popular even in the nineteenth century.

<table>
<thead>
<tr>
<th>Projection</th>
<th>Parallels</th>
<th>Meridians</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common name</td>
<td>Shape</td>
<td>Equidistant at</td>
</tr>
<tr>
<td>Bacon</td>
<td>straight boundary meridians</td>
<td>circular</td>
</tr>
<tr>
<td>Apian 1</td>
<td>straight central meridian</td>
<td>circular</td>
</tr>
<tr>
<td>Apian 2</td>
<td>straight central meridian</td>
<td>elliptical</td>
</tr>
<tr>
<td>Fournier 1</td>
<td>circular boundary and central meridians</td>
<td>elliptical</td>
</tr>
<tr>
<td>Fournier 2</td>
<td>straight boundary meridians</td>
<td>elliptical</td>
</tr>
</tbody>
</table>
General features of basic globular projections

Oval and Extended Globular Maps

Nicolosi circular boundary and central circular Equator meridians

Ortelius map, approximate extension of first Apian design; central meridian 0°

Even though no globular projection was intended for showing more than one hemisphere at a time, all can be extended for spanning the whole world. In the examples, the central circular hemisphere is identical to that of the original projection, except for the central meridian.

Oval maps have straight parallels and simple curves for meridians. The maps by Agnese (ca. 1540) and Ortelius (1570) are probably derived from Apian's first design, using semicircular arcs of fixed radius for the outer hemisphere. The result fits a 2:1 frame and looks like the modern projections III and IV by Eckert but it is not, of course, equal-area. Neither it is pseudocylindrical because meridian spacing is not uniform in the inner and outer hemispheres.

The second design by Apian uses elliptical meridians and, as a modern curiosity, can be extended using arcs of inverse excentricity. Despite being pseudocylindrical, passing resemblance to Mollweide's elliptical map is only superficial.

Second Apian map extended to whole world, central meridian 0°
A similar construction applied to Nicolosi's globular map produces an apple-shaped frame, an interesting contrast to Van der Grinten's IV projection.

Octant Projections

Resembling star projections, octant maps were briefly used contemporaneously to early oval maps. They divide the Earth surface in eight equal-shaped pieces, usually bounded by the Equator and four meridians. Each spherical triangle is separately projected into a roughly triangular octant; if its edges are circular arcs centered on the opposite vertex of an equilateral triangle, the octant's shape is called a Reuleaux triangle.

Leonardo da Vinci presented a Reuleaux triangle-based octant world map (ca. 1514) with gores arranged in separate, shamrock-like hemispheres, but omitting the graticule; the projection method can only be speculated.
O. Finé published a partial graticule (1551), again without specifying details. Quite possibly, parallels are nonconcentric circular arcs equally spaced along both meridian edges and central meridian of each octant; meridians are equally spaced along the Equator.

**Projections for Navigators and Radio Operators**

Besides their pedagogical value, many map projections are invaluable for specialized professionals. For instance, a common problem is finding the shortest route across the Earth surface between two points. Such path is always part of a geodesic or great circle on the globe surface. The geodesic is used by ship and aircraft navigators attempting to minimize distances, while radio operators with directional antennae look for a bearing yielding the strongest signal.

**The Cylindrical Conformal Projection**

**Mercator's Projection**

The great Flemish cartographer Gerhard Kremer became famous with the Latinized name Gerardus Mercator. A revolutionary invention, the cylindrical projection bearing his name has a remarkable property: any straight line between two points is a loxodrome, or line of constant course on the sphere. In the common equatorial aspect, the Mercator loxodrome bears the same angle from all meridians. In other words, if one draws a straight line connecting a journey's starting and ending points on a Mercator map, that line's slope yields the journey direction, and keeping a constant bearing is enough to get to one's destination.
The only conformal cylindrical projection, Mercator's device was a boon to navigators from the 16th-century until the present, despite suffering from extreme distortion near the poles: Antarctica is enormously stretched, and Greenland is rendered about nine times larger than actual size. Indeed, stretching grows steadily towards the top and bottom of the map (in the equatorial form, in higher latitudes; the poles would be actually placed infinitely far away). Like all conformal projections, Mercator's was not intended for worldwide wall maps.

Although important, a Mercator map is not the only one used by navigators, as the loxodrome does not usually coincide with the geodesic, except in short travels.

This projection was possibly first used by Etzlaub ca. 1511; however, it was for sure only widely known after Mercator's atlas of 1569. Mercator probably defined the graticule by geometric construction; E. Wright formally presented equations in 1599.

**Transverse Mercator Projection**

More commonly applied to large-scale maps, the transverse aspect preserves every property of Mercator's projection, but since meridians are not straight lines, it is better suited for topography than navigation.

Equatorial, transverse and oblique maps offer the same distortion pattern.

The transverse aspect, with equations for the spherical case, was presented by Lambert in his seminal paper (1772). The ellipsoidal case was developed, among others, by the great mathematician Carl Gauss.
(ca. 1822) and by Louis Krüger (ca. 1912); it is frequently called the Gauss conformal or Gauss-Krüger projection.

The UTM Grid

The best known use of the transverse Mercator projection is the specialized form called Universal Transverse Mercator (UTM) projection system.

The UTM defines a grid covering the world between parallels 84°N and 80°S. The grid is divided in sixty narrow zones, each centered on a meridian. Zones are identified by consecutive numbers, increasing from west to east (the first zone, immediately east of the 180° meridian, is numbered 1; zone 31 lies just east of the Greenwich meridian). A set of parallels divides the grid in rows, labeled by letters from C to X (I and O are not used) starting south. Therefore each zone comprises 20 quadrangles, identified by a number-letter pair. Quadrangles are in turn further subdivided in squares 100 km-wide, identified by double letter combinations.

Although no Mercator map is created by a perspective process, the cylinder is a useful visualization aid. The blue strip is zone 13 of the UTM grid; it is part of a cylindrical slice, approximating a spherical lune 6° wide at the equator and clipped by the 84°N and 80°S parallels.

Each zone is separately projected using the ellipsoidal form of the transverse Mercator projection with a secant case: scale of the central meridian is reduced by 0.04%, so two lines about 1°37" east and west of it have true scale. The UTM grid was designed for large-scale topographic mapping in separate sheets, not for whole world maps. In particular, sheets from different zones don't juxtapose exactly.
Within each quadrangle, any point may be located by its distance in meters, east from the central meridian and north from the Equator. The central meridian's coordinate is always 500,000; the Equator's coordinate is designated 0 for quadrangles in the northern hemisphere, and 10,000,000 for quadrangles in the southern hemisphere. Since the distance from poles to Equator is approximately 10,000 km, such offset origins ensure coordinates (called **false eastings** and **false northings**) are always positive.

The UTM grid is fairly regular, with a few exceptions:

- the polar cap south of 80°S is mapped by the ellipsoidal case of the **azimuthal stereographic** projection, and comprises semicircular "zones" A (west of Greenwich meridian) and B
- likewise, the cap north of 84°N is covered by an azimuthal stereographic projection comprising "zones" Y (west of Greenwich) and Z
- all quadrangles span 8° in the south-north direction, except those in row X, which ranges from 72°N to 84°N
- all quadrangles span 6° from west to east, except a few at rows X and V immediately east of the prime meridian

The original UTM system was adopted by the U.S. Army in 1949, and variations afterwards by several agencies throughout the world. Despite the name, it is not actually "universal" in the sense that each grid may be based on a different datum, so sheets from different grid sets may or may not be compatible. UTM maps are of course conformal, and distance and area distortion are limited by the large scale of individual sheets.

**Azimuthal Equidistant Projection**

Part of the **azimuthal** family and commonly used in the polar aspect, the **azimuthal equidistant** projection is invaluable for drawing some **geodesics** as straight lines. It also shows distances in true **scale** over the Earth surface (the captain of a nuclear submarine could use this projection to check which cities lie inside its destructive range).
This azimuthal equidistant map is centered around Campinas, Brazil. This kind of map requires a fairly dense geographical database as its periphery is strongly stretched.

Unfortunately, such properties are valid for the central point only, and each azimuthal equidistant map must be tailor-made for that specific location.

On the other hand, a gnomonic map does not preserve distances and cannot show the whole Earth, but it does map all geodesics to straight lines. In other words, there is no optimal single map which can show the shortest path between any two points on Earth.

As a rule, azimuthal projections make straightforward finding true directions from a single point: a radio operator whose hardware is stationed at the map's center could point its antenna for maximum gain towards anywhere on Earth.

Since the equidistant azimuthal preserves radial distances from the central point, here we immediately see that a shortest hypothetical flight from Campinas to Central Australia passes directly south over the pole.

Some Applications for Projections

Introduction
Frequently a proper choice of projection, scale, orientation and coordinates is crucial for a map to convey its message quickly and unequivocally, as demonstrated by a few examples.

When a reference is presented, the maps in this section are not scannings or reproductions but rather approximate reconstructions of the original
sources (for instance, the original faunal regions map does not use an equal-area projection) for illustration purposes.

**Faunal Regions and Global Connections**

For zoological classification purposes, the world is divided in *faunal regions.*

Here (after Oliver L. Austin and Arthur Singer, *Birds of the World*, Hamlyn Publ. 1968), a map based on Maurer's star projection presents all lands minus Antarctica. Since climate is a great determinant of faunistic features, most regional borders follow latitudes and are, therefore, roughly concentric in polar-aspect maps.

A land arrangement similar to the previous starlike map is employed in the latest version of Buckminster Fuller's *icosahedral* projection. It was used for presenting fiber cable routes and intercontinental network traffic in Thomas B. Allen's *The Future is Calling*, National Geographic 200(6), December 2001. The projection choice was appropriate, since connections to Antarctica are negligible and no routing lines had to be broken, even across oceans.

**Migration Paths**
Possible migration of early human population, according to the "Out of Africa" hypothesis; clipped Mercator map, central meridian 150°E

According to the "Out of Africa" theory, modern man appeared as a single African species nearly 100,000 years ago, then spread throughout the world (K. Wong, *Is Out of Africa Going Out the Door?*, Scientific American 281(2), August 1999). A simple coordinate rotation avoids cutting the migration path between Asia and Alaska.

Present and fossil teeth suggest several migration waves in the past, when reduced sea levels created bridges between now isolated Japanese and Aleutian islands. Cassini projection.

Some particular migration paths relate populations in eastern Asia which may have later populated Polynesia, northern Asia and the Americas. Dental anthropology (Christy G. Turner II, *Teeth and Prehistory in Asia*, Scientific American 260 (2), February 1989) provides evidence depending on several genetic factors, not culturally acquired and seldom affected by the environment.

**Orbital Tracks**
The crash of the Mir space station on the Pacific Ocean attracted a lot of attention. Most media coverage used cylindrical projections to depict the station's last moments following an apparently sinuous curve. Of course, the actual path of a low-orbit object looks much more simple when presented tridimensionally or as an azimuthal orthographic map:

Despite popular conceptions and illustrations, many satellites cruise at relatively low orbits, nearly grazing Earth's top atmosphere: a typical average altitude for Mir and the International Space Station is 390 km, while the Space Shuttle routinely performs orbits below 250 km (geostationary satellites ride much higher, at nearly 36000 km). The example is simplified, since orbiting objects neither follow exactly circular trajectories nor keep constant altitude; besides, the planet itself does not lie still but rotates beneath their paths.

The curve separating day/night areas is also roughly a great circle, thus the sinuous graphics in popular desktop programs displaying a "sun clock" on rectangular maps.

**Heavenly Maps**

For most of mankind history, stars and planets were mainly an aid for navigation and time reckoning. In this role, it is perfectly appropriate thinking of heavenly bodies as attached to an enormous spherical shell slowly rotating around Earth. Such a sphere can be mapped by exactly the same processes as used for ordinary maps. In fact, some map projections were first used by early astronomers for charting the stars instead of lands.
Two polar azimuthal stereographic maps present the night sky as seen in the north and south poles. Viewers in other latitudes and dates will see the constellations more or less tilted. In the boreal (northern) sky map, the Big Dipper appears at the 9 o'clock position, at the second innermost circle. The bright star at 10 o'clock is Arcturus; the Polar star is slightly below the heavenly North pole. In the austral (southern) sky, the two bright stars at 2 o'clock on the second innermost circle belong to Centaurus and point to the Southern Cross, slightly below and to the right; Sirius is the brightest star near the bottom center.

Summary
Several approaches attempt to classify projections. Most are orthogonal, thus any single projection may belong simultaneously to different categories. In others, like biology-inspired Maurer's, a branching taxonomy is applied.

Please note that, for ease of definition and visualization, some categories are informally described below in terms of parallels and meridians, thus some properties depend on the particular aspect used for the map. For instance, the coordinate lines in cylindrical maps cross at right angles in equatorial, but not in polar or oblique maps, although all other properties still hold; after all, the coordinate grid is only a set of conventional lines.

### Projections Classified by Geometry

<table>
<thead>
<tr>
<th>Category</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Azimuthal</td>
<td>Also called zenithal. Show true directions from single tangent point; in polar aspects all parallels are circular, and meridians straight lines uniformly spaced; unclipped map is circular. Projection can usually be defined by tangent point at a plane and location of light source (projection center)</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>Use a cylinder as intermediate projection surface; in equatorial aspect all parallels are straight horizontal, all meridians are straight vertical and uniformly spaced; unclipped map is rectangular</td>
</tr>
<tr>
<td>Conic</td>
<td>Use a conic intermediate projection surface; in polar aspect all parallels are concentric arcs of circle, all meridians are straight lines perpendicular to every parallel; unclipped maps are circular sectors</td>
</tr>
<tr>
<td>Pseudocylindrical</td>
<td>In equatorial aspect all parallels are straight horizontals; meridians are arbitrary curves, equally spaced along</td>
</tr>
</tbody>
</table>

The world according to Lambert's azimuthal equal-area map, in the equatorial aspect
every parallel

**Pseudoconic**
In polar or equatorial aspects all parallels are circular arcs, while meridians are arbitrary curves.

**Arbitrary**
Parallels and meridians are arbitrary curves; usually no purely geometric construction is defined. Some authors call "arbitrary", "conventional" or "compromise" any projection not derived from geometric devices, but custom-fit to a purpose.

In a sense, the cone includes as extreme cases both the cylinder (a cone with vertex at the infinite) and the plane (a cone with zero height). Therefore, the conic group generalizes the azimuthal and cylindrical and, broadly, pseudocylindrical and pseudoconic projections. Also, some consider a **polyconic** group to include projections where parallels are derived from circles, including modified azimuthals like Hammer's and Aitoff's. Actually, many so-called "azimuthal", "conic" or "cylindrical" projections are not built on a pure projective process using solids, but are so classified due to geometrical properties of the mapped coordinate grid.

Also, a **projective, geometric or perspective** projection can be described in exact analogy to a geometric set-up of light rays connecting the original surface to the map surface. Some authors call other projections "mathematical".

### Projections Classified by Property

<table>
<thead>
<tr>
<th>Category</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Equal-area</strong></td>
<td>Any region in the map has area linearly proportional to the corresponding region on the sphere; also called <strong>equivalent</strong> or <strong>authalic</strong>. Generally more useful for geographical comparisons and didactic purposes.</td>
</tr>
<tr>
<td><strong>Equidistant</strong></td>
<td>On the map there are two sets of points $A$ and $B$, such that, along a selected set of lines (not necessarily straight), distances from any point in $A$ to another in $B$ are proportional to the distances between corresponding points on the sphere, again along those corresponding lines. In other words, scale is constant on those lines, which are called <strong>standard</strong>. Most projections have such sets but few are actually called &quot;equidistant&quot;.</td>
</tr>
<tr>
<td><strong>Conformal</strong></td>
<td>In any small region of the map, two concurrent lines have the same angle as corresponding lines on the sphere, thus shapes are locally preserved. Also called <strong>orthomorphic</strong> or <strong>autogonal</strong>. Most important for navigational purposes.</td>
</tr>
<tr>
<td><strong>Aphylactic</strong></td>
<td>Some authors use this name for those projections which are neither conformal nor equivalent.</td>
</tr>
</tbody>
</table>
## Projections in a Nutshell

<table>
<thead>
<tr>
<th>Sample graticule</th>
<th>Common names</th>
<th>Main Features</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="azimuthal orthographic" /></td>
<td>azimuthal orthographic</td>
<td>Azimuthal, &quot;realistic&quot; view of Earth as seen from space</td>
</tr>
<tr>
<td><img src="image" alt="azimuthal stereographic" /></td>
<td>azimuthal stereographic</td>
<td>Azimuthal, conformal, circle-preserving; shows at most a hemisphere</td>
</tr>
<tr>
<td><img src="image" alt="gnomonic, central" /></td>
<td>gnomonic, central</td>
<td>Azimuthal, all great circles map to straight lines; extreme distortion far from the center; shows less than one hemisphere</td>
</tr>
<tr>
<td><img src="image" alt="azimuthal equidistant, zenithal equidistant" /></td>
<td>azimuthal equidistant, zenithal equidistant</td>
<td>Azimuthal, distances from center preserved, navigational purposes</td>
</tr>
<tr>
<td><img src="image" alt="Lambert's azimuthal equal-area" /></td>
<td>Lambert's azimuthal equal-area</td>
<td>Only possible azimuthal equal-area</td>
</tr>
<tr>
<td><img src="image" alt="globular projections by Bacon, Apian, Fournier, Ortelius, Nicolosi" /></td>
<td>globular projections by Bacon, Apian, Fournier, Ortelius, Nicolosi</td>
<td>General class of arbitrary projections bounded by a hemisphere in a circle; graticule comprising simple curves</td>
</tr>
<tr>
<td><img src="image" alt="Lambert's equal-area cylindrical; variations by Behrmann, Trystan Edwards, Gall (isographic), Peters, Dyer" /></td>
<td>Lambert's equal-area cylindrical; variations by Behrmann, Trystan Edwards, Gall (isographic), Peters, Dyer</td>
<td>Unique possible cylindrical equal-area projection, including scaled variants like Gall's and Hobo-Dyer</td>
</tr>
<tr>
<td><img src="image" alt="Gall's stereographic cylindrical" /></td>
<td>Gall's stereographic cylindrical</td>
<td>Neither conformal nor equal-area</td>
</tr>
<tr>
<td><img src="image" alt="Braun's stereographical cylindrical" /></td>
<td>Braun's stereographical cylindrical</td>
<td>Neither conformal nor equal-area</td>
</tr>
<tr>
<td><img src="image" alt="central cylindrical, centrographic cylindrical" /></td>
<td>central cylindrical, centrographic cylindrical</td>
<td>Neither conformal nor equal-area; not to be confused with Mercator's. Transverse aspect is the Wetch projection</td>
</tr>
</tbody>
</table>
equirrectangular, equidistant cylindrical, plain chart, plane chart; special case is simple cylindrical or plate carrée; Cassini (transverse aspect)

Cylindrical, very fast and easy to compute; in the most common case maps into a rectangle with aspect ratio 2 : 1 (twice wide as tall)

Mercator, cylindrical conformal; transverse ellipsoidal form called Gauss conformal or Gauss-Krüger

Only possible conformal cylindrical projection; foundation of the UTM grid

Miller

Cylindrical, arbitrary compromise to Mercator

Mollweide, elliptical, Babinet, homographic, homalographic

Pseudocylindrical, equal-area, meridians are ellipses; full map bounded by 2 : 1 ellipse; sometimes interrupted; variations include Atlantis

Sanson-Flamsteed, sinusoidal, Mercator equal-area

Pseudocylindrical, equal-area, meridians are sinusoids, parallels are equally spaced and standard lines; 2 : 1

Collignon

Pseudocylindrical, equal-area, meridians are straight lines. Two main variants, with triangular frame or symmetrical diamond with meridians broken at Equator

Goode homolosine

Pseudocylindrical, equal-area, hybrid joining Mollweide at poles, Sanson-Flamsteed at the equatorial band, almost always interrupted

Boggs eumorphic

Pseudocylindrical, equal-area, arithmetic average of Mollweide and Sanson-Flamsteed projections. Usually interrupted

Flat polar quartic

Pseudocylindrical, equal-area, poles are 1/3 as long as the Equator
Eckert I

Pseudocylindrical, 2 : 1, poles are half as long as the Equator, meridians are straight lines broken at Equator. Parallels equally spaced.

Eckert II

Pseudocylindrical, equal-area, 2 : 1, poles are half as long as the Equator, meridians are straight lines broken at Equator.

Eckert III

Pseudocylindrical, 2 : 1, meridians are elliptical arcs (boundary is circular). Parallels are equally spaced.

Eckert IV

Pseudocylindrical, equal-area, 2 : 1, meridians are elliptical arcs, circular at boundary.

Eckert V

Pseudocylindrical, 2 : 1, meridians are sinusoids, parallels are equally spaced. Particular case of Winkel's first projection

Eckert VI

Pseudocylindrical, equal-area, 2 : 1, poles are half as long as the Equator, meridians are sinusoids.

Robinson, orthophanic

Pseudocylindrical, compromise. Neither conformal nor equal-area

Winkel I

Pseudocylindrical (generalizes Eckert V), averages Sanson-Flamsteed and equidistant cylindrical, meridians are sinusoids

Winkel II

Pseudocylindrical, averages equidistant cylindrical and a modified elliptical projection

Pseudo-Eckert

Pseudocylindrical, equal-area, meridians are sinusoids

Quartic authalic

Pseudocylindrical, equal-area, meridians are 4th order polynomials
<table>
<thead>
<tr>
<th>Projection</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equidistant conic</td>
<td>Conic, constant meridian scale; limiting cases are azimuthal equidistant and cylindrical equidistant projections. Many variations, mostly in choice of standard parallels (Murdock, Euler). Others include de l'Isle's coniclike projection.</td>
</tr>
<tr>
<td>Braun stereographic conic</td>
<td>Conic, semicircular shape</td>
</tr>
<tr>
<td>Albers equal-area conic</td>
<td>Conic, equal-area; limiting cases are Lambert's equal-area conic and cylindrical projections.</td>
</tr>
<tr>
<td>Lambert's equal-area conic</td>
<td>Conic, equal-area; limiting case of Albers's conic, with a pole as standard parallel</td>
</tr>
<tr>
<td>Lambert's conformal conic, orthomorphic conic</td>
<td>Conic, conformal; limiting cases are azimuthal stereographic and Mercator projections</td>
</tr>
<tr>
<td>Polyconic, American Polyconic</td>
<td>Polyconic, parallels are nonconcentric arcs of circle with correct scale. Neither conformal nor equal-area.</td>
</tr>
<tr>
<td>Rectangular Polyconic, War Office</td>
<td>Polyconic, parallels are nonconcentric circular arcs crossing all meridians at right angles; either Equator or two parallels have correct length. Neither equivalent nor conformal</td>
</tr>
<tr>
<td>Aitoff</td>
<td>Stretching of modified equatorial azimuthal equidistant map; boundary is 2 : 1 ellipse</td>
</tr>
<tr>
<td>Wagner IX, Aitoff-Wagner</td>
<td>Modified Aitoff projection; neither equal-area nor conformal</td>
</tr>
<tr>
<td>Hammer, Hammer-Aitoff, Aitoff-Hammer</td>
<td>Modified from azimuthal equal-area equatorial map; equal-area, boundary is 2 : 1 ellipse; variations include Briesemeister, oceanic and</td>
</tr>
</tbody>
</table>
Briesemeister
Equal-area, simple oblique stretching of Hammer projection

Eckert-Greifendorff
Rescaled modification of Hammer projection. Equal-area

Winkel Tripel
Arithmetic mean of Aitoff and equidistant cylindrical projections

Stabius-Werner I
Pseudoconic, equal-area, parallels are equally spaced circular arcs centered on a pole

Werner, Stabius-Werner II, cordiform
Pseudoconic, equal-area, parallels are equally spaced circular arcs and standard lines, centered on a pole

Stabius-Werner III
Pseudoconic, equal-area, parallels are equally spaced circular arcs centered on a pole

Bonne
Pseudoconic, equal-area, parallels are equally spaced circular arcs and standard lines. General case of both Werner and sinusoidal

Peirce Quincuncial
World map in a square, central hemisphere in an inner square. Conformal except at edge midpoints. Other aspects by Guyou and Adams

Guyou
World map in 2:1 rectangle. Conformal except at hemisphere corners. Other aspects by Peirce and Adams

Adams's hemispheres on squares
Hemispheres in two squares. Conformal except at square corners. Other aspects by Guyou and Peirce

Adams's world on a square
Poles at opposite vertices; Equator at a diagonal. Conformal except at four vertices
<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adams's world on a square</td>
<td>Poles at midpoints of opposite edges. Conformal except at poles and four vertices</td>
</tr>
<tr>
<td>Conformal maps on polygons by Lee and Adams</td>
<td>Several conformal maps in polygons and polyhedra. Usually nonconformal at polygon vertices</td>
</tr>
<tr>
<td>Xarax's world in half a hexagon</td>
<td>Three-lobed world map. Conformal except at North pole and meridian breaks at each lobe</td>
</tr>
<tr>
<td>Eisenlohr</td>
<td>Fully conformal, no singular points. Scale constant along boundary. Optimal range of scale distortion for a conformal design</td>
</tr>
<tr>
<td>August, August epicycloidal</td>
<td>Conformal everywhere, with no singular points. Map bounded by a epicycloid. Base for Spilhaus's oceanic map</td>
</tr>
<tr>
<td>&quot;Lagrange&quot;</td>
<td>Map is bounded by a circle; meridians and parallels are circular arcs, except central meridian and Equator. Conformal except at the poles</td>
</tr>
<tr>
<td>Van der Grinten, Van der Grinten I</td>
<td>Boundary is a circle, meridians and parallels are circular arcs, except central meridian and Equator</td>
</tr>
<tr>
<td>Van der Grinten III</td>
<td>Boundary is a circle, meridians (except central) are circular arcs; parallels are horizontal lines intersecting central meridian at same points as in Van der Grinten I</td>
</tr>
<tr>
<td>Van der Grinten IV</td>
<td>Bounded by two intersecting circles, meridians are arcs of circle equally spaced along Equator, parallels are arcs of circle</td>
</tr>
<tr>
<td>Maurer's full-globular</td>
<td>Meridians along lines of Van der Grinten’s IV, outer meridians bounded by half limiting circles. Parallels are</td>
</tr>
<tr>
<td>Graticule</td>
<td>Description</td>
</tr>
<tr>
<td>---------------------------------------------------------------------------</td>
<td>---------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td><strong>Jäger Star</strong></td>
<td>Graticule comprising only straight lines. Eight unequal lobes, each symmetrical in core and outer hemisphere. Parallels linearly spaced in each lobe.</td>
</tr>
<tr>
<td><strong>Petermann Star</strong></td>
<td>Parallels are concentric, equally spaced arcs of circle, meridians are straight lines (most broken at the Equator). Neither conformal nor equal-area. Sometimes described with unequal lobes.</td>
</tr>
<tr>
<td><strong>Berghaus Star</strong></td>
<td>Five-lobed version of Petermann's projection</td>
</tr>
<tr>
<td><strong>Conoalactic</strong></td>
<td>Very similar to Berghaus, but northern hemisphere is based on equidistant conic; not to be confused with Cahill's &quot;butterfly&quot; map</td>
</tr>
<tr>
<td><strong>Maurer's S233</strong></td>
<td>Graticule comprises straight lines, with constant spacing. Neither conformal nor equal-area. Symmetrical case of Jäger's projection.</td>
</tr>
<tr>
<td><strong>Maurer's S231 (equal-area star)</strong></td>
<td>Parallels are concentric arcs of circle; northern hemisphere is a Lambert azimuthal map. Southern meridians are curved. Equal-area.</td>
</tr>
<tr>
<td><strong>&quot;Tetrahedral&quot;</strong></td>
<td>Central core is part of azimuthal equidistant hemisphere. Outer lobes are modified Werner with expanded parallel scaling. Neither conformal, equal-area nor polyhedral.</td>
</tr>
<tr>
<td><strong>Leonardo da Vinci's octant map</strong></td>
<td>Octant map, bound by circular arcs. Neither conformal nor equal-area.</td>
</tr>
<tr>
<td>Projection System</td>
<td>Description</td>
</tr>
<tr>
<td>--------------------------------------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Armadillo (orthoapsidal on torus)</td>
<td>Intermediate projection surface is torus with radii 1 and 1; final map is projected orthographically</td>
</tr>
<tr>
<td>Orthoapsidal on ellipsoid</td>
<td>Intermediate projection surface is ellipsoid; final map is projected orthographically</td>
</tr>
<tr>
<td>Arden-Close</td>
<td>Arithmetical mean of equal-area cylindrical map and its transverse aspect; neither equal-area nor conformal.</td>
</tr>
<tr>
<td>Trapezoidal</td>
<td>Pseudocylindrical, meridians are straight lines, sometimes symmetrically broken at Equator</td>
</tr>
<tr>
<td>Steve Waterman's projection system</td>
<td>Based on a truncated octahedron defined by the centers of packed spheres. Neither conformal nor equal-area.</td>
</tr>
</tbody>
</table>

**Resources**

**Links**

**Reference**

- Paul B. Anderson's huge [Gallery of Map Projections](#) has no descriptions, but probably the broadest range of publicly available projections, with more than 300 PDF maps.
- The [Rocky Mountain Mapping Center](#) at the USGS has a good collection of links.
- John Savard's introduction to [map projections](#) is very comprehensive. Don't overlook the many other interesting resources in his [home page](#).
- Birger Nielsen has a section on [maps](#); the [map projection](#) page has several examples with accompanying Perl code. (if you like tea, don't miss the rest of the site)
- Dr.-Ing. Rolf Böhlm's [site](#) presents many, many projections. It also describes an Assembly-like projection definition language and its interpreter.
- Professor Waldo Tobler's [site](#) includes several papers and presentations. It shows cartography can be both serious and fun.
- **Steve Waterman**'s description of his polyhedral map based on sphere-packing
- Dr. Mark R. Calabretta's papers on celestial and world coordinates, including applications of map projections to astronomy
- Karen Mulcahy's Map Projection Home Page, - at present, apparently rarely updated

**Map Suppliers**

- **ODT** publishes Waterman's and several other maps.
- **Art Source International** sells rare maps and prints.
- The **Buckminster Fuller Institute**, source of information and online store for *Dymaxion™* maps

**Software and Data**

- **MicroCAM**, a mapping package for MS-Windows by Scott Loomer
- Texture maps ("skins") for planets at **NASA**
- Mitchell Charity's **GIMP** script for generating gore maps from texture images
- Dave **Pape**'s collection of **Earth map** textures

**Bibliography**

The interested reader can find further reference material here:

**General Cartography**


**Projections**
• American Cartographic Association, *Which Map is Best?*, American Congress on Surveying and Mapping, 1986
  Short and practical booklet about the several conflicting requirements of an adequate map. Several projections are presented at identical scales for easy comparison of distortion patterns.

  Brief review of cartographic concepts, plus a few unusual projections.

• Snyder, John P., *Flattening the Earth*, University of Chicago Press, 1993.

  Detailed descriptions of map projections employed at the famous government mapping agency. Appendices include forward and inverse equations, besides interpolation tables.

**Software**
The following programs and data were used when composing these map projection pages:

• SG, my simple projection application, used for all outline and textured maps presented here
• GIMP, the GNU Image Manipulation program, for image retouching and conversion
• POV-Ray, a ray-tracer for solid modeling
• Data for heavenly maps available in coordinate form at the Flight Gear open-source flight simulator project, extracted from the xephem program

Formulas for the Eisenlohr and Van der Grinten IV projections were kindly donated by J.P. Snyder and Paul B. Anderson.

**And now something which did not fit anywhere else**
I hope this document has helped the reader to develop an interest in cartography, understand some important concepts and how to best choose (or refuse) a map projection. If not, one may always take comfort in the immortal stanzas of Lewis Carroll’s *The Hunting of the Snark - an Agony in Eight Fits*:

**Fit the First - The Landing**
...Navigation was always a difficult art,
    Though with only one ship and one bell:
And he feared he must really decline, for his part,
    Undertaking another as well
...

Fit the Second - The Bellman's Speech

...He had bought a large map representing the sea,
    Without the least vestige of land:
And the crew were much pleased when they found it to be
    A map they could all understand.

"What's the good of Mercator's North Poles and Equators,
    Tropics, Zones, and Meridian Lines?
So the Bellman would cry: and the crew would reply
    "They are merely conventional signs!

"Other maps are such shapes, with their islands and capes!
    But we've got our brave Captain to thank"
(So the crew would protest) "that he's bought us the best -
    A perfect and absolute blank!"
...